An extended Monge-Ampère operator

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Goal: define a $(dd^c u)^p$ with nice properties for larg class of psh fcns

Background:

Cplx MA operator: $u \mapsto (dd^c u)^p$ def if u C^2, ≥ 0 if u psh

Bedford-Taylor 80's

- If u psh locally bdd & T pos closed current then $dd^c u \wedge T := dd^c(uT)$ well-def pos closed
  ⤳ can define $(dd^c u)^p := dd^c(u (dd^c u)^{p-1})$

Continuity: if $u_j \downarrow u$ then $(dd^c u_j)^p \to (dd^c u)^p$

Unbdd psh?

- Demailly: BT def if u bdd outside small set
- Blocki & Cegrell $\Omega \Subset \mathbb{C}^n$
  $D(\Omega) = \{u \in psh(\Omega)$ st $\forall u_j \downarrow u$ (smooth psh) $(dd^c u_j)^n \to \mu =: (dd^c u)^n\}$

Non-pluripolar MA (BT, BEGZ 2010)

$\langle dd^c u\rangle^p := \lim_{\ell \to \infty} \mathbb{1}_{\{u > -\ell\}} (dd^c \max(u, -\ell))^p$

- pos, closed if $\langle dd^c u\rangle^p$ loc. finite
- not continuous for $u_j \downarrow u$

Ex $u = \log|f|^2$ f holo fcn in $\Omega \subseteq \mathbb{C}^n$

$dd^c u = [f=0]$ ← current of integration (↑ Poincaré-Lelong)

$\langle dd^c u\rangle = 0$

New class & new MA

$\Omega \Subset \mathbb{C}^n$

- Def: • $u \in psh(\Omega)$ is in $\mathcal{G}(\Omega)$ if $\forall p \leq n$ $u \langle dd^c u\rangle^{p-1}$ locally finite
  • $u \in \mathcal{G}(\Omega)$: $[dd^c u]^p := dd^c(u \langle dd^c u\rangle^{p-1})$
  $S_p(u) := [dd^c u]^p - \langle dd^c u\rangle^p$

Prop: closed pos (p,p)-currents

Ex $u = \log|f|^2 + v$ ← smooth psh $\in \mathcal{G}(\Omega)$

↖ holo fcn in $\mathbb{C}^2$

$[dd^c u] = dd^c u = \underbrace{[f=0]}_{S_1(u)} + \underbrace{dd^c v}_{\langle dd^c u\rangle}$

$[dd^c u]^2 = dd^c(u\, dd^c v) = [f=0] \wedge dd^c v + (dd^c v)^2$

Ex (AW 14, A 05)

$u \in psh(\Omega)$ w analytic singularities $\in \mathcal{G}(\Omega)$

(i.e. locally $u = c \log|f|^2 + b$ ← loc. bdd, $c > 0$, f tuple of holo fcns)

Pf: Hironaka ⤳ Ex above

In part: $\exists\, u \in \mathcal{G}(\Omega) \setminus D(\Omega)$

Ex $u = -(-\log|f|^2)^\varepsilon \in \mathcal{G}$ if $\varepsilon < 1/2$ $\forall n$

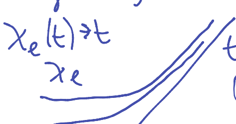
$u \langle dd^c u\rangle$ $\quad u \langle dd^c u\rangle^p$

Continuity

Thm A: $u \in \mathcal{G}(\Omega)$

Then $\lim_{\ell \to \infty} (dd^c \max(u, -\ell))^p = [dd^c u]^p$

- More generally, if χ_ℓ convex, bdd from below st $\chi_\ell(t) \nearrow t$ then $(dd^c(\chi_\ell \circ u))^p \to [dd^c u]^p$



[A-Blocki-W] def if u anal sing

Note $u_\ell = \max(u, v - \ell)$ smooth psh

Thm B: $u \in \mathcal{G}(\Omega)$

$(dd^c u_\ell)^p \to [dd^c u]^p + \sum_{j=1}^{p-1} (dd^c v)^{p-j} S_j(u)$

Rmk: If $S_j(u) \neq 0$ some $j < n$ then $u \notin D(\Omega)$

Global version

(X, ω) cpt Kähler mfd

φ is ω-psh if $dd^c\varphi + \omega \geq 0$

⤳ can define $\mathcal{G}(X, \omega)$

$\varphi \in \mathcal{G}(X,\omega)$ if $\underbrace{\varphi + g}_{\in \mathcal{G}}$, where $dd^c g = \omega$,

⤳ can define $[dd^c\varphi + \omega]^p := [dd^c(\varphi + g)]^p$

Thm C $\varphi \in \mathcal{G}(X, \omega)$ Then

$(dd^c \max(\varphi, -\ell) + \omega)^p \to [dd^c\varphi + \omega]^p + \sum_{j=1}^{p-1} \omega^{p-j} \wedge S_j^\omega(\varphi)$

(Blocki) def if φ anal sing

Thm: $\int_X \langle dd^c\varphi + \omega\rangle^n + \sum_{j=1}^{n} \int_X \omega^{n-j} \wedge S_j^\omega(\varphi) = \int_X \omega^n$

Ex $\mathcal{E}_1(X, \omega) \subset \mathcal{G}(X, \omega)$

$E_k^{np}(\varphi) = \frac{1}{k+1} \sum_{j=1}^{k} \int_X \varphi \langle dd^c\varphi + \omega\rangle^j \omega^{n-j}$

kth non-pluripolar energy

$\mathcal{G}(X, \omega) \supseteq \{\varphi \in PSH(X, \omega) : E_{n-1}^{np}(\varphi)$ finite$\}$

$$\int_X [dd^c(\varphi+\omega)]^n + \sum_{j=1}^{n-1} \int \omega^{n-j} \wedge S_j^{\omega}(\varphi) = \int_X \omega^n$$