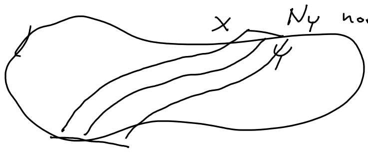
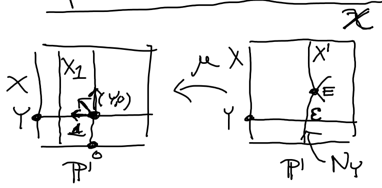
Deforming Kähler manifolds to normal bundles

X cplx mfd of dim n, $Y \subseteq X$ subm.



Deformation to the normal cone



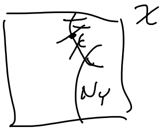
$X' \cong Bl_Y X$, E exc. div.

$\mathcal{E} \cong \mathbb{P}(N_Y \oplus \mathbb{C}) = N_Y \cup E$

Let (X,ω) cpt Kähler, $\mathcal{X}$ also cpt Kähler

Def: A Kähler form Ω on $\mathcal{X}$ s.t. $\Omega|_{X_1} = \omega$ will be called a (standard) Kähler def. on (X,ω) to N_Y.

Observation:



Blowing up things in $\mathcal{X}'$ does not change N_Y

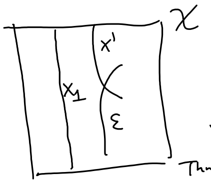
Def: A pair $(\mathcal{X}', \Omega)$ where $\mathcal{X}'$ is a smooth modification of $\mathcal{X}$ with center contained in X' and Ω Kähler form on $\mathcal{X}'$ s.t. $\Omega|_{X_1} = \omega$ will be called a (nonstd.) Kähler def. of (X,ω) to N_Y.

Thm 1 (–): $\forall \varepsilon > 0$ $\exists$ (nonstd.) Kähler def $(\mathcal{X}', \Omega)$ s.t. $\int_{N_Y} \Omega^n > (1-\varepsilon)\int_X \omega^n$

Def:
- A closed (1,1)-current T on X is Kähler current if $T \geq \delta\omega$
- $T = dd^c u + \Theta$ has anal. sing. if $u \underset{\text{l.c.}}{\equiv} \alpha \log(\Sigma |f_i|^2) + g$, g C^∞, $E_T := \{f_i = 0\}$

The algebraic case $\gamma = c_1(L)$ was proved by Boucksom–Jasson–Favre and Lazarsfeld–Mustață '09.

How to get Thm 1 from Thm 2?



$\{X_1\} = \{\mathcal{E}\} + \{X'\}$

$\beta = (\pi_X \circ \mu)^* \alpha + b\{X_\infty\} - c\{\mathcal{E}\}$

$b > c$: β big

$\mathrm{vol}_{\mathcal{X}|X_1}(\beta) = \boxed{\int_X \omega^n}$

Thm 2 $\;\Vert$

$\boxed{\mathrm{vol}_{\dots}(\beta)} + \boxed{\mathrm{vol}_{\mathcal{X}|X'}(\beta)} \leftarrow$ small!

Note that $\int_{X_\tau} \Omega^n$ ind. of τ $\Rightarrow \int_{N_Y} \Omega^n = \int_X \omega^n - \int_{X'} \Omega^n$ ← volume loss

Question: Can we make the volume loss arb. small?

Answer: No, not always

Let $\alpha := [\omega]$, $\beta := [\Omega]$, then since $\Omega|_{X_1} = \omega$:

$\beta = (\pi_X \circ \mu)^* \alpha + b\{X_\infty\} - c\{\mathcal{E}\}$, $b, c \in \mathbb{R}$, and

$\beta|_{X'} = (\mu')^* \alpha - c\{E\}$, $\mu' : X' \to X$ blowup of X along Y

Def: $\varepsilon(\alpha, Y) := \sup\{t : (\mu')^*\alpha - t\{E\} \text{ is Kähler}\}$

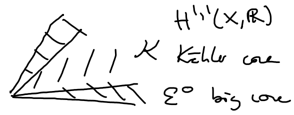
Seshadri const.

Note: β Kähler $\Rightarrow$ $c < \varepsilon(\alpha, Y)$

$\int_{X'} \Omega^n = \int_{X'} \beta^n = \int_{X'} ((\mu')^* - c\{E\})^n \geq \int_{X'} ((\mu')^* - \varepsilon(\alpha,Y)\{E\})^n$ ← can be > 0!

Ex: $X = \mathbb{P}^1 \times \mathbb{P}^1$, $Y = \{(0,0)\}$, $\omega = \omega_{FS} \oplus \omega_{FS}$

- $\gamma \in H^{1,1}(X, \mathbb{R})$ is big if it contains a Kähler current



- volume of γ: $\mathrm{vol}(\gamma) := \sup_{\substack{T \in \gamma \\ \text{Kähler curr.} \\ \text{w. anal. sing.}}} \int_{X \setminus E_T} T^n$

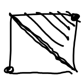
- $Z \subseteq X$ subm.: restricted volume $\mathrm{Vol}_{X|Z}(\gamma) := \sup_{\substack{T \in \gamma \\ \dots}} \int_{Z \setminus E_T} T^m$

Transcendental ~~version~~ analog of theory of big l.b.
(Demailly, Boucksom ...)

Thm 2 (–): If γ big and Z smooth hypersurface ($\not\subseteq E_{nK}(\gamma)$) then

$\frac{d}{dt}\Big|_{t=0} \mathrm{vol}(\gamma + t\{Z\}) = n\,\mathrm{vol}_{X|Z}(\gamma)$. In part.

$\mathrm{vol}_{X|Z}(\gamma)$ only depends on $\{Z\}$.



$\exists\, T \in \beta$ $\int_{N_Y} T^n > (1-\varepsilon)\int_X \omega^n$

$\mathcal{X}'$ resolution of sing. of T

find $\Omega|_{X_1} = \omega$ by gluing ...

$$\mathrm{vol}_{X|X_1}(\beta) = \int_X \omega^{n}$$

Thm 2

$\{X_1\} = \{E\} + \{X'\}$

$$\boxed{\mathrm{vol}_{X|E}(\beta)} + \boxed{\mathrm{vol}_{X|X'}(\beta)} \leftarrow \text{small!}$$

$$\mathrm{vol}_{X|X'}(\beta) \leq \mathrm{vol}(\beta_{|X'}) = \mathrm{vol}((\mu')^{*}\alpha - c\{E\}) \to 0$$

$c \to \mathrm{psef}(\alpha, Y)$

(Boucksom)