Estimates for collapsing families of complex Monge-Ampère equations

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Analysis of Monge-Ampère, a tribute to Ahmed Zeriahi

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based on joint works with Hans-Joachim Hein and with Simion Filip

Collapsing families of complex Monge-Ampère equations

$f: X^{m+n} \to B^m$ surjective holomorphic map with connected fibers, $m, n > 0$

(X, ω_X) compact Kähler manifold, (B, ω_B) compact Kähler reduced irreducible analytic space

$f^*\omega_B$ smooth semipositive definite closed real $(1,1)$-form with $(f^*\omega_B)^{m+n} = 0$

Given $F \in C^\infty(X)$, for each $t \geqslant 0$ by Yau's Theorem there is a unique smooth φ_t solving

$$(f^*\omega_B + e^{-t}\omega_X + i\partial\overline{\partial}\varphi_t)^{m+n} = c_t e^{-nt} e^F \omega_X^{m+n}, \quad \int_X \varphi_t \omega_X^{m+n} = 0, \quad f^*\omega_B + e^{-t}\omega_X + i\partial\overline{\partial}\varphi_t > 0$$

where $c_t \in \mathbb{R}_{>0}$ is defined by integrating, and is bounded away from 0 and ∞ uniformly in t.

Question

What happens to φ_t as $t \to \infty$?

Uniform boundedness

$$(f^*\omega_B + e^{-t}\omega_X + i\partial\overline{\partial}\varphi_t)^{m+n} = c_t e^{-nt} e^F \omega_X^{m+n}, \quad \int_X \varphi_t \omega_X^{m+n} = 0, \quad f^*\omega_B + e^{-t}\omega_X + i\partial\overline{\partial}\varphi_t > 0$$

Conjecture (Tian 07)

$\sup_X |\varphi_t| \leqslant C$, *for all* $t \geqslant 0$.

Proved by Kołodziej-Tian 07 in some special cases.

Theorem (Demailly-Pali, Eyssidieux-Guedj-Zeriahi 07)

Conjecture is true, and C *depends only on the background geometry and on* $\|e^F\|_{L^p}, p$ *for any* $p > 1$.

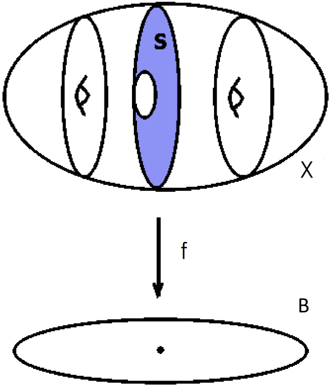
Where does this equation come from?

Fibration structure

$f : X^{m+n} \to B^m$ as before

$D \subset B$ critical values of f, $S = f^{-1}(D) \subset X$,
$f : X \backslash S \to B \backslash D$ proper holomorphic submersion

Fibers $X_z = f^{-1}(z)$ with $z \in B \backslash D$ are compact Kähler n-folds, pairwise diffeomorphic, in general not biholomorphic



Many examples are constructed when X projective, $L \to X$ holomorphic line bundle with $\int_X c_1(L)^{m+n} = 0, c_1(L) \neq 0$ and L semiample, taking f to be the map given by $|kL|$ for some $k \gg 1$ sufficiently divisible

Collapsing of Kähler metrics

• **Calabi-Yau metrics:** X^{m+n} is a compact Kähler manifold with $c_1(X) = 0$ in $H^2(X, \mathbb{R})$ (Calabi-Yau), and take ω_X Ricci-flat Kähler and $F = 0$. The fibers X_z are Calabi-Yau.

Then $\omega_t := f^*\omega_B + e^{-t}\omega_X + i\partial\overline{\partial}\varphi_t$ is exactly the unique Ricci-flat Kähler metric on X cohomologous to $f^*\omega_B + e^{-t}\omega_X$

• **Kähler-Ricci flow:** X^{m+n} is a compact Kähler manifold with K_X semiample and $\kappa(X) = m$, and f the map given by $|kK_X|$ for some $k \gg 1$ sufficiently divisible. The fibers X_z are Calabi-Yau.

In this case the Monge-Ampère equations are

$$((1 - e^{-t})f^*\omega_B + e^{-t}\omega_X + i\partial\overline{\partial}\varphi_t)^{m+n} = e^{-nt}e^{F+\varphi_t+\dot{\varphi}_t}\omega_X^{m+n},$$

with F suitably chosen, then $\omega_t := (1 - e^{-t})f^*\omega_B + e^{-t}\omega_X + i\partial\overline{\partial}\varphi_t > 0$ solves the Kähler-Ricci flow

$$\frac{\partial}{\partial t}\omega_t = -\mathrm{Ric}(\omega_t) - \omega_t$$

Collapsing away from the singular fibers

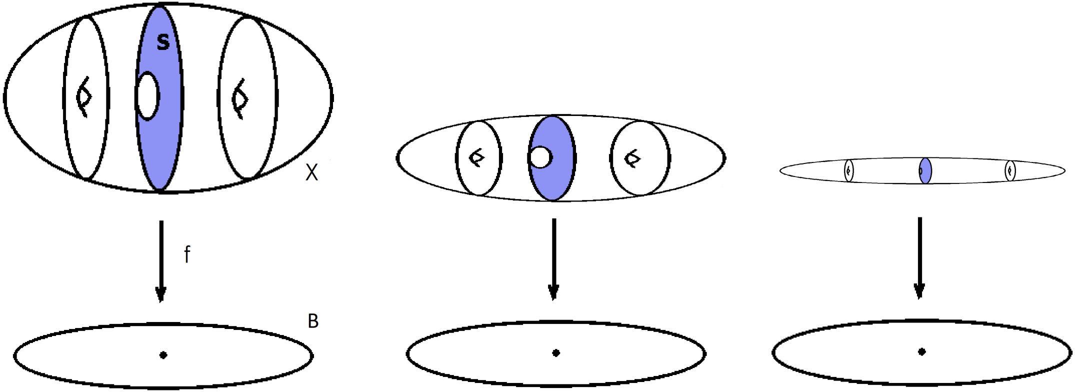
Theorem (T. 09)

Assume the Calabi-Yau setting. Then there is $\omega_\infty = \omega_B + i\partial\overline{\partial}\varphi_\infty$ *Kähler on* $B\backslash D$, $\mathrm{Ric}(\omega_\infty) = \omega_{WP} \geqslant 0$, *such that* $\omega_t \to f^*\omega_\infty$ *weakly and* $\varphi_t \to f^*\varphi_\infty$ *in* $C^{1,\alpha}_{\mathrm{loc}}(X\backslash S), 0 < \alpha < 1$. *For every* $K \Subset X\backslash S$ *we have*

$$C^{-1}(f^*\omega_B + e^{-t}\omega_X) \leqslant \omega_t \leqslant C(f^*\omega_B + e^{-t}\omega_X)$$

on K *for all* $t \geqslant 0$.

- Uses crucially the L^∞ bound
- $\omega_{WP} \equiv 0 \Leftrightarrow$ all $X_z, z \in B\backslash D$ are isomorphic
- Follows from Gross-Wilson 00 for elliptic $K3$ with 24 singular fibers
- Same result for the Kähler-Ricci flow (Song-Tian 06, Fong-Zhang 12), where $\mathrm{Ric}(\omega_\infty) = -\omega_\infty + \omega_{WP}$
- Di Nezza-Guedj-Guenancia 20: Log Calabi-Yau setting (with singularities)


S
X
f
B

Main Conjecture

Conjecture

X^{m+n} Calabi-Yau, $f : X \to B$ fibration as before. $\omega_t = f^*\omega_B + e^{-t}\omega_X + i\partial\overline{\partial}\varphi_t$ Ricci-flat Kähler metrics and ω_∞ their weak limit. Then as $t \to \infty$

$$\omega_t \to f^*\omega_\infty$$

in $C^k_{\mathrm{loc}}(X\backslash S, g_X)$ for all $k \geqslant 0$.

Previous results

- (Gross-Wilson 00) Elliptic $K3$ with 24 singular fibers, via gluing construction
- (Fine 04) $S = \emptyset$
- (Gross-T.-Zhang 11, Hein-T. 14) X_z tori
- (T.-Weinkove-Yang 14) $k = 0$
- (Hein-T. 18) f isotrivial
- (Hein-T. 18) $\omega_t \to f^*\omega_\infty$ in $C^\alpha_{\mathrm{loc}}(X\backslash S, g_X), 0 < \alpha < 1$
- (Li 18, Chen-Viaclovsky-Zhang 19) other gluing constructions $m = 1, n = 1, 2$

Main Theorem

Theorem (Hein-T. 21)

The conjecture is true.

Our higher-order estimates are local on the base (away from D), so we may suppose that B is just a ball in $\mathbb{C}^m$. Then $f^{-1}(B) \cong (B \times Y, J)$ where J is a fiber-preserving complex structure which need not be product. Fischer-Grauert 65: J is a product $\Leftrightarrow$ fibers X_z mutually biholomorphic

Our result uses crucially that the fibers of f are compact without boundary, and the theorem fails in general without this.
For $z \in B \backslash D$ let $\omega_{F,z} = \omega_X|_{X_z} + i\partial\overline{\partial}\rho_z$ Ricci-flat on X_z, $\int_{X_z} \rho_z \omega_X^n = 0$ and

$$\omega_F = \omega_X + i\partial\overline{\partial}\rho$$

semi-Ricci-flat form. In general $\omega_F \not\geq 0$ on $X \backslash S$ (Cao-Guenancia-Păun-T. 18)

$$\omega_t^{\mathrm{ref}} = f^*\omega_\infty + e^{-t}\omega_F$$

Kähler metric on $K \Subset X \backslash S$ for $t \gg 0$, $\omega_t = \omega_t^{\mathrm{ref}} + i\partial\overline{\partial}\psi_t$, $\psi_t \to 0$ in $C^{1,\alpha}_{\mathrm{loc}}$

Special cases

Theorem

Suppose either X_z are mutually biholomorphic (Hein-T. 18) or X_z are tori (Gross-T.-Zhang 11, Hein-T. 14). Then, up to shrinking B, we have

$$\sup_{B\times Y} |\nabla^{k,g_t^{\mathrm{ref}}}\omega_t|_{g_t^{\mathrm{ref}}} \leqslant C_k,$$

for all $t \geqslant 0, k \geqslant 0$.

Recall that are X_z mutually biholomorphic iff J is a product, and then ω_t^{ref} are product Kähler metrics on $B \times Y$.

Uniform C^k estimates with respect to the shrinking metrics g_t^{ref} are much stronger than what the main conjecture/theorem requires.

Shrinking Hölder norms

In general, we have written $\omega_t = \omega_t^{\mathrm{ref}} + i\partial\overline{\partial}\psi_t$, $\psi_t \to 0$ in $C^{1,\alpha}_{\mathrm{loc}}$

It is tempting to try to show that ψ_t is uniformly bounded in $C^k(\omega_t^{\mathrm{ref}})$ for all k, like in the two special cases. First, since now ω_t^{ref} is not a product metric, we work instead in our C^∞ trivialization $f^{-1}(B) \cong B \times Y$, and for each $z \in B$ we consider the Riemannian product metrics $g_{z,t} = g_\infty + e^{-t} g_{F,z}$ and $g_t = g_{0,t}$

Using the family of Levi-Civita connections ∇^z of $g_{z,t}$, define a connection $\mathbb{D}$, define $C^k(g_t)$ norms using $\mathbb{D}^k$, and define Hölder seminorms

$$[\mathbb{D}^j \tau]_{C^\alpha(B\times Y, g_t)} = \sup_{x,x'} \frac{|\mathbb{D}^j\tau(x) - \mathbb{P}_{x'x}\mathbb{D}^j\tau(x')|_{g_t}}{d^{g_t}(x,x')^\alpha}$$

Then hope to show that $i\partial\overline{\partial}\psi_t$ is bounded in $C^{k,\alpha}(g_t)$ (analog of the earlier special cases). This turns out to be true for $k = 0, 1$ but **false** for $k \geqslant 2$!

Asymptotic Expansion I

Theorem (Hein-T. 21)

We have

$$\omega_t = f^*\omega_\infty + e^{-t}\omega_F - e^{-2t} i\partial\overline{\partial}\Delta_Y^{-1}\Delta_Y^{-1}(g_\infty^{\mu\overline{\nu}}(\langle A_\mu, \overline{A_\nu}\rangle - \omega_{\mathrm{WP},\mu\overline{\nu}})) + \mathrm{error},$$

where the fiber-restriction of error is $O(e^{-(2+\beta)t})$ in $C^{2,\beta}$ for some $\beta > 0$.

Here $A_\mu = T^{1,0}X_z$-valued $(0,1)$-form on X_z, harmonic wrt $\omega_{F,z}$, representing Kodaira-Spencer class of variation of complex structure of X_z in the direction $\frac{\partial}{\partial z^\mu}$

When X_z tori or mutually isomorphic then that term vanishes and the error decays faster than e^{-Nt} for all N.

In general the e^{-2t} term is nonzero, already for threefolds fibered by $K3$ surfaces over $\mathbb{P}^1$. Its $C^{2,\alpha}(g_t)$ norm then blows up, in general.

Asymptotic Expansion II

Theorem (Hein-T. 21)

Given $0 \leqslant j \leqslant k, 0 < \alpha < 1$ *and* $z \in B$ *there is a ball* $B' = B_r(z) \subset B$ *and smooth function* $G_{i,p,k}, 2 \leqslant i \leqslant j, 1 \leqslant p \leqslant N_{i,k}$, *on* $B' \times Y$ *so that*

$$\omega_t = f^*\omega_\infty + e^{-t}\omega_F + \gamma_{t,0} + \gamma_{t,2,k} + \cdots + \gamma_{t,j,k} + \eta_{t,j,k}$$

$$\gamma_{t,0} = i\partial\overline{\partial}\underline{\psi_t} \to 0 \textit{ in } C^j(B')$$

$$\gamma_{t,i,k} = i\partial\overline{\partial}\sum_{p=1}^{N_{i,k}} \mathfrak{G}_{t,k}(A_{t,i,p,k}, G_{i,p,k}) \to 0 \textit{ in } C^j(B' \times Y, g_X)$$

$$\eta_{t,j,k} = \textit{remainder} \to 0 \textit{ in } C^j(B' \times Y, g_t^{\mathrm{ref}})$$

$$A_{t,i,p,k} \textit{ functions on } B' \to 0 \textit{ in } C^{j+2}(B')$$

All objects are also bounded in $C^{\bullet,\alpha}$

$\mathfrak{G}_{t,k}$: approximate Green operator, $\Delta^{g_t^{\mathrm{ref}}}\mathfrak{G}_{t,k}(A, G) \approx AG$ for A polynomial of degree $< 2k+2$

$A_{t,i,p,k} \approx$ fiberwise L^2 component of $\mathrm{tr}^{g_t^{\mathrm{ref}}}\eta_{t,i-1,k}$ onto $\mathbb{R}.G_{i,p,k}$

Structure of proof

The proof goes by fixing k and induction on $0 \leqslant j \leqslant k$ and consists of two parts. First, we construct the pieces $\gamma_{t,0}, \gamma_{t,2,k}, \cdots, \gamma_{t,j,k}$ of the expansion. For this the key part is the construction of the obstruction functions $G_{i,p,k}$, which arise as obstructions to the remainder $\eta_{t,i-1,k}$ being bounded in $C^{i,\alpha}(g_t)$. Once we have these we recursively define

$$A_{t,i,p,k} = n(\mathrm{pr}_B)_*(G_{i,p,k}\eta_{t,i-1,k} \wedge \omega_F^{n-1}) + e^{-t}\mathrm{tr}^{\omega_\infty}(\mathrm{pr}_B)_*(G_{i,p,k}\eta_{t,i-1,k} \wedge \omega_F^{n})$$

$$\gamma_{t,0} = i\partial\overline{\partial}\underline{\psi_t}, \ \gamma_{t,i,k} = i\partial\overline{\partial}\sum_{p=1}^{N_{i,k}} \mathfrak{G}_{t,k}(A_{t,i,p,k}, G_{i,p,k}),$$

$$\eta_{t,0,k} = \omega_t - \omega_t^{\mathrm{ref}} - \gamma_{t,0}, \ \eta_{t,i,k} = \eta_{t,i-1,k} - \gamma_{t,i,k}, i \geqslant 2$$

$$\mathfrak{G}_{t,k}(A, G) = \sum_{\ell=0}^{k}(-1)^\ell e^{-\ell t}\Delta_{\mathbb{C}^m}^\ell A \, \Delta_Y^{-\ell-1} G, \ \text{in product case}$$

$$\omega_t = f^*\omega_\infty + e^{-t}\omega_F + \gamma_{t,0} + \gamma_{t,2,k} + \cdots + \gamma_{t,j,k} + \eta_{t,j,k}$$

The second part shows by contradiction and blowup that $\gamma_{t,0}, \eta_{t,j,k}$ and $A_{t,i,p,k}$ satisfy the desired estimates.

Ideas

In the blowup argument, assuming the limit is $\mathbb{C}^m \times Y$, the main issue is when we try to improve regularity by linearizing the Monge-Ampère equation

$$(\omega_t^{\mathrm{ref}} + \gamma_{t,0} + \gamma_{t,2,k} + \cdots + \gamma_{t,j,k} + \eta_{t,j,k})^{m+n} = c_t e^{-nt} \omega_\infty^m \wedge \omega_F^n$$

$$\mathrm{tr}^{g_t^{\mathrm{ref}}}(\gamma_{t,0} + \gamma_{t,2,k} + \cdots + \gamma_{t,j,k} + \eta_{t,j,k}) + (\mathrm{nonlinearities}) = c_t e^{-nt} \frac{\omega_\infty^m \wedge \omega_F^n}{(\omega_t^{\mathrm{ref}})^{m+n}} - 1$$

Split all objects into j-jets at the blowup basepoint (ill-behaved, but polynomials) plus Taylor remainders (well-behaved). Move the jets into ω_t^{ref} and linearize wrt this new metric. Then the linear term has good convergence, the nonlinearities are small, but RHS (mixture of background data and jets of the solution) does not satisfy good bounds. A posteriori it has to converge to some limit, necessarily of the form

$$K_0(z) + \sum_{q=1}^{N} K_q(z) H_q(y)$$

where K_q polynomials of degree $\leqslant j$ and H_q's in $\mathcal{G}$ =fiberwise linear span of $G_{i,p,k}, 2 \leqslant i \leqslant j$.

Ideas

By construction, in the linear term the part with Taylor remainders of $\gamma_{t,2,k}, \ldots, \gamma_{t,j,k}$ lies in $\mathcal{G}$, while the part with Taylor remainders of $\gamma_{t,0}, \eta_{t,j,k}$ lies in $\mathcal{G}^\perp$, since

$$\mathrm{tr}^{g_t^{\mathrm{ref}}} \gamma_{t,i,k} = \Delta^{g_t^{\mathrm{ref}}} \sum_{p=1}^{N_{i,k}} \mathfrak{G}_{t,k}(A_{t,i,p,k}, G_{i,p,k})$$

is close to the fiberwise L^2 component of $\mathrm{tr}^{g_t^{\mathrm{ref}}} \eta_{t,i-1,k}$ onto $\oplus_p \mathbb{R}.G_{i,p,k}$.
Setting this equal to

$$K_0(z) + \sum_{q=1}^{N} K_q(z) H_q(y)$$

we conclude that in the limit the Taylor remainders of $\gamma_{t,2,k}, \ldots, \gamma_{t,j,k}$ are polynomials of degree $\leqslant j$ (hence zero), and those of $\gamma_{t,0}, \eta_{t,j,k}$ have trace equal to a polynomial of degree $\leqslant j$, contradiction to linear Liouville Theorem.

Collapsing Monge-Ampère equations without fibration structure

X^{m+n} compact Calabi-Yau manifold, α closed real $(1,1)$-form such that $[\alpha] \in H^{1,1}(X,\mathbb{R})$ is nef, and suppose $[\alpha^m] \neq 0$ in $H^{2m}(X,\mathbb{R})$ but $[\alpha^{m+1}] = 0$ in $H^{2m+2}(X,\mathbb{R})$, with $m, n > 0$.

Then $[\alpha] + e^{-t}[\omega_X]$ is a Kähler class for all $t \geqslant 0$, let ω_t be the unique Ricci-flat metric in this class

We can write $\omega_t = \alpha + e^{-t}\omega_X + i\partial\overline{\partial}\varphi_t$, $\int_X \varphi_t \omega_X^{m+n} = 0$, and again we have

$$(\alpha + e^{-t}\omega_X + i\partial\overline{\partial}\varphi_t)^{m+n} = c_t e^{-nt}\omega_X^{m+n}$$

Conjecture (T.)

(a) *We have* $\sup_X |\varphi_t| \leqslant C$, *for all* $t \geqslant 0$.

(b) *We have higher order estimates for* φ_t *on compact subsets away from a closed analytic subset of* X

By work of Fu-Guo-Song 17, conjecture (a) is equivalent to saying that every nef class on a Calabi-Yau manifold contains a closed positive current with bounded potentials. The Calabi-Yau assumption is necessary.

Results on K3 surfaces

Theorem (Filip-T. 17,18)

Conjecture (b) is false: there are projective $K3$ surfaces X with α as above, such that ω_t does not converge locally uniformly away from any closed analytic subset.

In these examples, the limit potential φ_∞ is $C^\gamma(X)$ for some $\gamma > 0$ (Dinh-Sibony 05).
Probably optimal.

Theorem (Filip-T. 21)

Conjecture (a) is true if X projective $K3$ surface with no (-2)-curves and $\mathrm{rkPic}(X) \geqslant 3$, and $[\alpha] \in NS(X, \mathbb{R})$ nef. Furthermore, there is $\alpha + i\partial\overline{\partial}\varphi \geqslant 0$ with $\varphi \in C^0(X)$, and φ be chosen to vary continuously as we vary $[\alpha]$ on the boundary of the ample cone, and the resulting map $[\alpha] \mapsto \alpha + i\partial\overline{\partial}\varphi$ is $\mathrm{Aut}(X)$-equivariant.

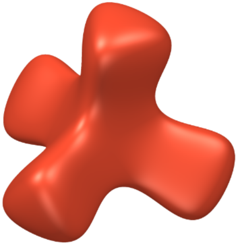
C^0 is probably optimal
Sibony-Verbitsky: if $[\alpha]$ is irrational then it contains a unique closed positive current

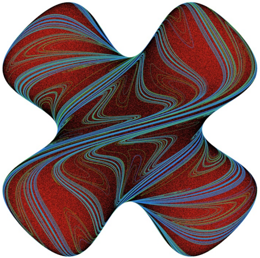
$(2,2,2)$ Examples (Wehler 88, Mazur, Silverman 92)

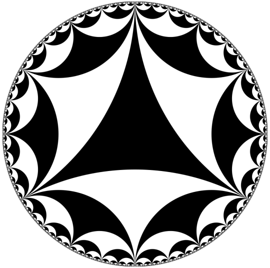
$X = \{P = 0\} \subset \mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1, \quad \deg P = (2,2,2),$ P generic

3 projections to $\mathbb{CP}^1 \times \mathbb{CP}^1$ exhibit X as a ramified $2:1$ cover, with involutions $\sigma_1, \sigma_2, \sigma_3$

$T = \sigma_1 \circ \sigma_2 \circ \sigma_3$ has infinite order, and "chaotic" behavior (positive entropy)







Dynamical nef classes

To disprove conjecture (b), take $T : X \to X$ as above, and $[\alpha]$ a nontrivial eigenvector of T^* on $H^{1,1}(X,\mathbb{R})$, with eigenvalue $\lambda > 1$

Cantat 99: $[\alpha]$ is nef, $\int_X \alpha^2 = 0$, $[\alpha] \in NS(X,\mathbb{R})$ and it contains a unique closed positive current η, $T^*\eta = \lambda\eta$

If conjecture is true then η is smooth on a Zariski open subset. But if this happens, a dynamical rigidity theorem of Cantat-Dupont 14 and Filip-T. 18 would imply that X Kummer, which is not the case.

To prove conjecture (a) in that case, use the Kawamata-Morrison Cone Conjecture (Thm of Sterk 85): there is a rational polyhedral fundamental domain for the action of $\mathrm{Aut}(X)$ on the nef cone in $NS(X,\mathbb{R})$. The quotient of the set of Kähler classes in $NS(X,\mathbb{R})$ with $\int_X [\omega]^2 = 1$ by $\mathrm{Aut}(X)$ is a finite-volume hyperbolic orbifold of dimension $\rho(X) - 1$. A positive current with C^0 potentials in $[\alpha]$ is constructed by following a hyperbolic geodesic that converges to $[\alpha]$, pulling back a Ricci-flat Kähler metric by a family $g_i \in \mathrm{Aut}(X)$ (and rescaling at each step), and carefully keeping track of the growth of g_i.

A different family of collapsing Ricci-flat metrics

$f: X \to \mathbb{P}^1$ an elliptically fibered $K3$ surface, and assume that there is a twist $T: X \to X$, i.e. a fiber-preserving automorphism of infinite order

Let $\alpha = f^*\omega_{FS}$, so $T^*\alpha = \alpha$. Fix a Ricci-flat metric ω, let

$$\omega_n = \frac{(T^n)^*\omega}{n^2},$$

then (up to an initial rescaling) $[\omega_n] \to [\alpha]$, tangentially to the boundary of the Kähler cone.

Question

What happens to ω_n as $n \to \infty$?

Letting $h = T^*\omega - \omega$ we can write $\omega_n = (1 - \frac{1}{n})\alpha + \frac{h}{n} + \frac{\omega}{n^2} + i\partial\bar{\partial}\varphi_n$

Theorem (Filip-T. 21)

We have $\sup_X |\varphi_n| \leqslant C$ and $\omega_n \to f^*(\omega_{FS} + i\partial\bar{\partial}u)$ weakly, where $\omega_{FS} + i\partial\bar{\partial}u \geqslant 0$ and $u \in L^\infty(\mathbb{P}^1) \cap C^\infty(\mathbb{P}^1 \backslash D)$. If the singular fibers of f are reduced and irreducible, then $u \in C^0(\mathbb{P}^1)$. The limit current $\omega_{FS} + i\partial\bar{\partial}u$ is in general different from ω_∞.

Thank you, and auguri to Ahmed!