The Iterated S^3 Sasaki Joins

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Joint work with Charles Boyer

Outline

Kähler Geometry

Kähler Geometry

- N is a smooth compact mani(orbi-)fold of real dimension $2d_N$.
- A Kähler structure is the quadruple (N, J, g, ω)
- $[\omega] \in H^2(N, \mathbb{R})$ is the **Kähler class**.
- the Riemannian **Ricci tensor** $r : TN \otimes TN \to C^\infty(N)$ gives us the **Ricci form**, $\rho(X, Y) = r(JX, Y)$.
- Scalar curvature is denoted by $Scal$.

Sasaki geometry

Sasaki geometry:

A Sasaki structure on a smooth compact manifold M of dimension $2n+1$ is defined by a quadruple $\mathcal{S} = (\xi, \eta, \Phi, g)$ where

- η is **contact 1-form** defining a subbundle (contact bundle) in TM by $\mathcal{D} = \ker \eta$.
- ξ is the **Reeb vector field** of η $[\eta(\xi) = 1$ and $\xi \rfloor d\eta = 0]$
- Φ is an endomorphism field which annihilates ξ and satisfies that $J = \Phi|_{\mathcal{D}}$ is a complex structure on the contact bundle $(d\eta(J\cdot, J\cdot) = d\eta(\cdot, \cdot))$
- $g := d\eta \circ (\Phi \otimes \mathbb{1}) + \eta \otimes \eta$ is a Riemannian metric
- ξ is a Killing vector field of g which generates a one dimensional foliation $\mathcal{F}_\xi$ of M whose transverse structure is Kähler.
- $(dt^2 + t^2 g, d(t^2\eta))$ is Kähler on $M \times \mathbb{R}^+$ with complex structure I: $IY = \Phi Y + \eta(Y) t\frac{\partial}{\partial t}$ for vector fields Y on M, and $I(t\frac{\partial}{\partial t}) = -\xi$.

Corresponding Kähler Foliations

- If ξ is **regular**, M has a free S^1 action generated by ξ and the quotient of the foliation $\mathcal{F}_\xi$ is compact Kähler manifold N whose Kähler cohomology class lies in $H^2(N,\mathbb{Z})$.
- If ξ is **quasi-regular**, M has a locally free S^1 action generated by ξ and the quotient of the quasi-regular foliation is a compact Kähler orbifold $\mathcal{Z}$ with finite cyclic local uniformizing groups whose orbifold Kähler cohomology class lies in $H^2_{orb}(\mathcal{Z},\mathbb{Z})$.
- If not regular or quasi-regular, it is **irregular**
- In the quasi-regular case, the orbifold Chern class $c_1^{orb}(\mathcal{Z})$ pulls back to the basic first Chern class $c_1(\mathcal{F}_\xi)$.

Transverse Homothety:

- If $\mathcal{S} = (\xi, \eta, \Phi, g)$ is a Sasakian structure, so is $\mathcal{S}_a = (a^{-1}\xi, a\eta, \Phi, g_a)$ for every $a \in \mathbb{R}^+$ with $g_a = ag + (a^2 - a)\eta \otimes \eta$.
- So Sasakian structures come in rays.

Deforming the Sasaki structure:

In its contact structure isotopy class:

- $\eta \to \eta + d^c\phi, \quad \phi \quad$ is basic
- ϕ is basic: $\xi \rfloor \phi = 0$ and $\mathcal{L}_\xi \phi = 0$
- This corresponds to a deformation of the transverse Kähler form $\omega_T \to \omega_T + dd^c\phi$ in its basic Kähler class (or induced Kähler class on the transverse mani-/orbifold in the regular/quasi-regular case).
- "Up to isotopy" means that the Sasaki structure might have been deformed as above.

In the Sasaki Cone:

- Pick maximal T^k, $0 \leq k \leq n+1$ in the Sasaki automorphism group $\mathfrak{Aut}(\mathcal{S}) = \{\phi \in \mathcal{D}iff(M) \,|\, \phi^*\eta = \eta, \, \phi^*J = J, \, \phi^*\xi = \xi, \, \phi^*g = g\}$.
- The unreduced Sasaki cone is $\mathfrak{t}^+ = \{\xi' \in \mathfrak{t}_k \,|\, \eta(\xi') > 0\}$
- Each element in $\mathfrak{t}^+$ determines a new Sasaki structure with the same underlying CR-structure.

Sasaki-Einstein metrics

Sasaki-Einstein metrics

- The Ricci tensor of g behaves as follows:
  - $r(X,\xi) = 2n\,\eta(X)$ for any vector field X
  - $r(X,Y) = r_T(X,Y) - 2g(X,Y)$, where X, Y are sections of $\mathcal{D}$ and r_T is the transverse Ricci tensor
- The scalar curvature of g behaves as follows: $Scal = Scal_T - 2n$
- If the transverse Kähler structure is Kähler-Einstein then we say that the Sasaki metric is η-Einstein.
- If η-Einstein and $Scal_T > 0$, then exactly one of the Sasaki structures in the η-Einstein ray is actually Sasaki-Einstein.
- If $\mathcal{S} = (\xi, \eta, \Phi, g)$ is Sasaki-Einstein, then we must have that $c_1(\mathcal{D})$ is a torsion class (for example, it vanishes).
- $\mathcal{S} = (\xi, \eta, \Phi, g)$ has constant scalar curvature (CSC) iff the transverse Kähler structure has constant scalar curvature.
- $\mathcal{S} = (\xi, \eta, \Phi, g)$ is η-Einstein (or CSC) iff its entire ray is η-Einstein (or CSC)

Generalized Orbifold Calabi construction

History:

The generalized Calabi construction was presented by Apostolov, Calderbank, Gauduchon, and T-F. Here we make a modest generalization, allowing for the base space to be a normal projective algebraic variety with cyclic orbifold singularities. Other than that, we are staying in the "admissible" niche of the construction. In this form, credit is due to Calabi, Guan, Hwang, Koiso, LeBrun, Pedersen, Poon, Sakane, Singer, Simanca, and others.

S^1-orbibundles (briefly)

S^1-orbibundles (briefly)

Assume that N is a normal (compact) projective algebraic variety with a fixed orbifold structure which we write as the pair (N, Δ_N) where Δ_N is a sum of irreducible branch divisors:

$$\Delta_N = \sum (1 - \frac{1}{m_j}) D_j$$

where m_j are the ramification indices, and $D_j \in \mathrm{Div}(N)$.
By Theorem 4.3.15 in Boyer and Galicki's book the isomorphism classes of S^1 orbibundles, M, over (N, Δ_N) are uniquely determined by their orbifold first Chern class denoted by $c_1^{orb}(M/N) \in H^2(N, \mathbb{Q})$. Furthermore, when M has a Sasakian structure, $c_1^{orb}(M/N)$ is an orbifold Kähler class with Kähler form denoted by ω_N.

Generalized Orbifold Calabi construction

The ingredients in the construction are as follows:

- [Base] A cyclic Kähler orbifold (N, Δ_N) equipped with a Kähler orbifold metric, g_{base}, whose Kähler form, $\omega_{base} = 2\pi\omega_N$, satisfies that $[\omega_N]$ is a primitive orbifold Kähler class. Note that a primitive orbifold class $[\omega_N]$ is obtained from a primitive integer class $[\omega_N]_I$, viz $[\omega_N] = \frac{[\omega_N]_I}{\Upsilon_N}$. ($\Upsilon_N$ is order of (N, Δ_N))
- [Fiber] A weighted projective line $(\mathbb{CP}^1_{\mathbf{m}} = \mathbb{CP}^1_{v^0,v^\infty}/\mathbb{Z}_m, g_{\mathbf{m}}, \omega_{\mathbf{m}})$ with orbifold Kähler structure $(g_{\mathbf{m}}, \omega_{\mathbf{m}})$. Here $(m^0, m^\infty) = m(v^0, v^\infty)$ and v^0, v^∞ are coprime.
- A principal S^1 orbi-bundle, $P_n \to (N, \Delta_N)$, with a principal connection of curvature $n\,\omega_{base} \in \Omega^{1,1}((N, \Delta_N), \mathbb{R})$, where S^1 acts on $\mathbb{CP}^1_{\mathbf{m}}$, $n \in \mathbb{Z} \setminus \{0\}$, and $\gcd(n, m) = 1$.
- A constant $0 < |r| < 1$ with the same sign as n

From this data we may define the orbifold

$$(S_n, \Delta_{\mathbf{m},N}) = P_n \times_{S^1} \mathbb{CP}^1_{\mathbf{m}} \stackrel{\pi^{orb}}{\longrightarrow} (N, \Delta_N)$$

$$\Delta_{\mathbf{m},N} = \Delta_{\mathbf{m}} + \pi^{-1}(\Delta_N) = (1 - \tfrac{1}{m^0})D^0 + (1 - \tfrac{1}{m^\infty})D^\infty + \sum(1 - \tfrac{1}{m_j})\pi^{-1}(D_j)$$

Where $\Delta_{\mathbf{m}}$ is a branch divisor consisting of the zero D^0 and infinity D^∞ sections of $S_n = \mathbb{P}(\mathbb{1} \oplus L_n)$ with ramification indices $\mathbf{m} = (m^0, m^\infty) = m(v^0, v^\infty)$. Moreover, $c_1^{orb}(L_n) = n[\omega_N]$. The choice of r from above ($0 < |r| < 1$ with the same sign as n), determines a so-called admissible Kähler class Ω_r.

Admissible Metrics:

On $(S_n, \Delta_{\mathbf{m},N})$ one can construct explicit orbifold Kähler metrics (g, ω) in Ω_r that each are determined by smooth functions $\Theta : (-1, 1) \to \mathbb{R}^+$ satisfying certain boundary conditions.

If g_{base} has constant scalar curvature we call such metrics admissible.

The point of this is:

Due to work by the of the people mentioned above, one can now write up the appropriate ODE for say admissible KE and CSC where KE comes with a further assumption that g_{base} is KE.

Remark:

The Calabi toric metrics by E. Legendre are special cases - see also recent work by Apostolov, Calderbank, Gauduchon, and Legendre.

Bott Manifolds and Orbifolds

Bott manifolds

First introduced by Grossberg and Karshon.

We call M_k the *stage k Bott manifold* of the *Bott tower of height n*:

$M_n \xrightarrow{\pi_n} M_{n-1} \xrightarrow{\pi_{n-1}} \cdots M_2 \xrightarrow{\pi_2} M_1 = \mathbb{CP}^1 \xrightarrow{\pi_1} pt,$

where M_k is defined inductively as the total space of the projective bundle $\mathbb{P}(\mathbb{1} \oplus L_k) \xrightarrow{\pi_k} M_{k-1}$ with fiber $\mathbb{CP}^1$, and some holomorphic line bundle L_k on M_{k-1}.Then $D_{u_i^0}$ and $D_{u_i^\infty}$ are the natural exceptional divisors coming from the zero and infinity sections at each stage and we denote the Poincaré duals in $H^2(M_n(A), \mathbb{Z})$ of the divisor classes $[D_{u_i^\infty}]$ and $[D_{u_i^0}]$ by x_i and y_i, respectively.

Consider the matrix $A = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ A_2^1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{n-1}^1 & A_{n-1}^2 & \cdots & 1 & 0 \\ A_n^1 & A_n^2 & \cdots & A_n^{n-1} & 1 \end{pmatrix}$, $\quad A_j^i \in \mathbb{Z}$. Then

$M_n = M_n(A)$ is the unique Bott tower for which the pullback of $c_1(L_k)$ to the total space is $\alpha_k = \sum_{j=1}^{k-1} A_k^j x_j$ for $k \in \{1, \ldots, n\}$.

Examples

- Stage 1, M_1 is just $\mathbb{CP}^1$
- Stage 2, If $A = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$ we get a Hirzebruch surface $M_2(a) = \mathbb{P}(\mathbb{1} \oplus \mathcal{O}(a)) \to M_1$.
- Stage 3, For $A = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix}$, we denote $M_3(A)$ by $M_3(a, b, c)$. It is the total space of a bundle $\mathbb{P}(\mathbb{1} \oplus \mathcal{L}) \to M_2(a)$ where b, c determines the Chern class of $\mathcal{L} \to M_2(a)$.

Bott Orbifolds

Bott Orbifolds

We construct Bott orbifold towers by putting an orbifold structure on the invariant divisors $D_{u_i^0}, D_{u_i^\infty}$ to make them branch divisors with ramification indices m_i^0, m_i^∞, respectively.
A Bott orbifold tower of height n is a Bott tower of orbifolds of the form

$$(M_n(A), \Delta^T_{\mathbf{m}_n}) \xrightarrow{\pi_n} (M_{n-1}(A), \Delta^T_{\mathbf{m}_{n-1}}) \xrightarrow{\pi_{n-1}} \cdots (M_2(A), \Delta^T_{\mathbf{m}_2}) \xrightarrow{\pi_2} (M_1 = \mathbb{CP}^1, \Delta_{\mathbf{m}_1}) \xrightarrow{\pi_1} (pt, \emptyset),$$

where $\Delta^T_{\mathbf{m}_n} = \sum_{i=1}^n \Delta_{\mathbf{m}_i} = \sum_{i=1}^n \Big(\big(1 - \frac{1}{m_i^0}\big) D_{u_i^0} + \big(1 - \frac{1}{m_i^\infty}\big) D_{u_i^\infty} \Big)$ and $(M_i, \Delta^T_{\mathbf{m}_i})$ is the log pair associated to the total space of the projective orbibundle

$$\mathbb{P}_{\mathbf{m}}(\mathbb{1} \oplus L_i) \xrightarrow{\pi_i} (M_{i-1}, \Delta^T_{\mathbf{m}_{i-1}})$$

with fiber $\mathbb{CP}^1(v_i^0, v_i^\infty)/\mathbb{Z}_{m_i}$ for some holomorphic line orbibundle L_i on M_{i-1} where $\mathbf{m}_i = (m_i^0, m_i^\infty) = m_i(v_i^0, v_i^\infty)$ with v_i^0, v_i^∞ relatively prime, so $m_i = \gcd(m_i^0, m_i^\infty)$.

The $S^3_{\mathbf{w}}$ join construction

History/Comment:

- Developed generally by (Boyer, Galicki, Ornea)
- is the analogue of Kähler products.

The ingredients in the construction are as follows:

- A compact quasi-regular Sasaki manifold, M, over (N, Δ_N) such that $c_1^{orb}(M/N) = [\omega_N] \in H^2(N, \mathbb{Q})$ is a primitive orbifold Kähler class.
- $c_1^{orb}(S^3_{\mathbf{w}}/\mathbb{CP}^1_{\mathbf{w}}) = [\omega_{\mathbf{w}}] = [\frac{\omega_{FS}}{w^0 w^\infty}]$, where ω_{FS} is the Kähler form of the standard Fubini-Study metric on $\mathbb{CP}^1$ such that $[\omega_{FS}] \in H^2(\mathbb{CP}^1, \mathbb{Z})$ is primitive and $\omega_{\mathbf{w}}$ is the transverse extremal Kähler form on $\mathbb{CP}^1_{\mathbf{w}}$ of the canonical extremal Sasaki structure on the weighted sphere $S^3_{\mathbf{w}}$.
- $w^\infty < w^0$ (for convenience)
- On $M \times S^3_{\mathbf{w}}$, $L = \frac{1}{2l^0}\xi_M - \frac{1}{2l^\infty}\xi_{\mathbf{w}}$ for a pair of relatively prime positive integers l^0, l^∞

- We form the (l^0, l^∞)-join of M and $S^3_{\mathbf{w}}$, $M_{\mathbf{l},\mathbf{w}} = M \star_{\mathbf{l}} S^3_{\mathbf{w}}$, by taking the quotient by the action induced by L:

$$
\begin{array}{ccc}
M \times S^3_{\mathbf{w}} & & \\
 & \searrow \pi_L & \\
\Big\downarrow \pi_2 & & M_{\mathbf{l},\mathbf{w}} \\
 & \swarrow \pi_1 & \\
(N, \Delta_N) \times \mathbb{CP}^1_{\mathbf{w}} & &
\end{array}
$$

- π_2 is the product of the projections of the standard Sasakian projections $\pi_M : M \longrightarrow (N, \Delta_N)$ and $S^3_{\mathbf{w}} \longrightarrow \mathbb{CP}^1_{\mathbf{w}}$.
- $M_{\mathbf{l},\mathbf{w}}$ is a S^1-orbibundle (generalized Boothby-Wang fibration).
- $M_{\mathbf{l},\mathbf{w}}$ has a natural quasi-regular Sasakian structure for all relatively prime positive integers l^0, l^∞. Fixing l^0, l^∞ fixes the contact orbifold.

Lemma (Using Proposition 7.6.6 in Boyer-Galicki book)

If $(M, \mathcal{S})$ is a smooth quasi-regular Sasaki manifold of order $\Upsilon_{\mathcal{S}}$, then the join $M_{\mathbf{l},\mathbf{w}} = M \star_{\mathbf{l}} S^3_{\mathbf{w}}$ is smooth if and only if $\gcd(l^\infty \Upsilon_{\mathcal{S}}, l^0 w^0 w^\infty) = 1$ where the order $\Upsilon_{\mathcal{S}}$ is precisely the order Υ_N of the quotient orbifold (N, Δ_N), i.e. $\Upsilon_{\mathcal{S}} = \Upsilon_N$.

The **w**-Sasaki cone

- The unreduced Sasaki cone $\mathfrak{t}^+_{M,\mathbf{l},\mathbf{w}}$ of the join $M_{\mathbf{l},\mathbf{w}}$ has a 2-dimensional subcone $\mathfrak{t}^+_{\mathbf{w}}$ is called the **w**-Sasaki cone.
- $\mathfrak{t}^+_{\mathbf{w}}$ is inherited from the Sasaki cone on S^3
- We are interested in quasi-regular Sasakian structures $\mathcal{S}$ in $\mathfrak{t}^+_{\mathbf{w}}$, that is, Reeb vector fields $\xi_{\mathbf{v}}$ that lie on the integer lattice $\Lambda_{\mathbf{w}} \subset \mathfrak{t}^+_{\mathbf{w}}$. They are completely determined by the pair (v^0, v^∞) of relatively prime positive integers.

Theorem (Boyer, T-F)

Let $(M, \mathcal{S})$ be a quasi-regular Sasakian structure and $M_{\mathbf{l},\mathbf{w}} = M \star_{\mathbf{l}} S^3_{\mathbf{w}}$ the $S^3_{\mathbf{w}}$-join of M. Then the quotient of $M_{\mathbf{l},\mathbf{w}}$ by the S^1 action generated by a quasi-regular Reeb vector field $\xi_{\mathbf{v}} \in \mathfrak{t}^+_{\mathbf{w}}$ is $(S_n, \Delta_{\mathbf{m},N}) \stackrel{\pi^{orb}}{\longrightarrow} (N, \Delta_N)$ from before, where

- $m = \frac{l^\infty}{s}$, $n = l^0 \frac{(w^0 v^\infty - w^\infty v^0)}{s}$, $s = \gcd(l^\infty, |w^0 v^\infty - w^\infty v^0|)$, $\gcd(m, n) = 1$.
- *The induced primitive (by rescale) orbifold Kähler form $\omega_{n,\mathbf{m},N}$ on the orbifold $(S_n, \Delta_{N,\mathbf{m}})$ satisfies $[\omega_{n,\mathbf{m},N}] = \frac{s}{4\pi \gcd(s\Upsilon_N, w^0 v^\infty l^0) m v^0 v^\infty} \Omega_r$, where Ω_r is the admissible Kähler class from above with $r = \frac{w^0 v^\infty - w^\infty v^0}{w^0 v^\infty + w^\infty v^0}$.*
- *Let $\mathcal{S}_{\mathbf{v}} = (\xi_{\mathbf{v}}, \eta_{\mathbf{v}}, \Phi, g_{\mathbf{v}})$ be a quasiregular Sasakian structure in the **w** cone $\mathfrak{t}^+_{\mathbf{w}}$. The order Υ of $\mathcal{S}_{\mathbf{v}}$ is the product $m v^0 v^\infty \Upsilon_N$.*

Connecting (hopefully) with E. Legendre's talk:

The orbifold-Kähler quotient of each quasi-regular ray given by a pair of co-prime (v^0, v^∞) is the *CR*-twist by a positive Killing potential f of the product Kähler quotient of the join with respect to the original ray (given by (w^0, w^∞)).

Up to homothety, $f = \mathfrak{z} + 1/r$, where we now view $\mathfrak{z}$ as the lift of the moment map of $(\mathbb{CP}^1[\mathbf{w}], \omega_{\mathbf{w}})$ to the product.

The Iterated Join

$S^3_{\mathbf{w}}$-iterated joins of height k:

$$M^{2k+1}_{\mathbf{l},\mathbf{w}} = M^{2k-1}_{\mathbf{l},\mathbf{w}} \star_{\mathbf{l}_{k-1}} S^3_{\mathbf{w}_k} = \big((S^3_{\mathbf{w}_1} \star_{\mathbf{l}_1} S^3_{\mathbf{w}_2}) \star_{\mathbf{l}_2} \cdots \star_{\mathbf{l}_{k-2}} S^3_{\mathbf{w}_{k-1}} \big) \star_{\mathbf{l}_{k-1}} S^3_{\mathbf{w}_k},$$

where at each stage k we choose a quasi-regular Sasakian structure given by a primitive (meaning the quotient Kähler class is primitive) quasi-regular Reeb field $\xi_{\mathbf{v}_j}$ in the subcone $\mathfrak{t}^+_{\mathbf{w}_k}$ and for each i, the components l_i^0, l_i^∞ and w_i^0, w_i^∞ are relatively prime. The latter condition ensures that our Sasaki orbifolds are simply connected.

Lemma

The (orbifold) order of a quasi-regular Sasakian structure $\mathcal{S}_k$ with Reeb field $\xi_{\mathbf{v}}$ on the height k iterated join is given by $\Upsilon_{\mathcal{S}_k} = m_k v_k^0 v_k^\infty \cdots m_2 v_2^0 v_2^\infty w_1^0 w_1^\infty = \prod_{i=2}^k \mathrm{lcm}(m_i^0, m_i^\infty) w_1^0 w_1^\infty$ *where* $m_i = \frac{l_{i-1}^\infty}{\gcd(l_{i-1}^\infty, |w_i^0 v_i^\infty - w_i^\infty v_i^0|)}$. *Further, assuming that the previous stages are smooth,* $M^{2k+1}_{\mathbf{l},\mathbf{w}}$ *is smooth if and only if* $\gcd(l_{k-1}^\infty \Upsilon_{\mathcal{S}_{k-1}}, l_{k-1}^0 w_k^0 w_k^\infty) = 1$.

Using induction and our previous theorem we now have

Theorem (Boyer, T-F)

Let $M^{2n+1}_{\mathbf{l},\mathbf{w}}$ *be an iterated* S^3*-join of height* n*. Then for each* $k \in \{2, \dots, n\}$ *the* S^1 *orbifold quotient of a quasi-regular Sasakian structure* $\mathcal{S}_{\mathbf{l},\mathbf{w},\mathbf{v}}$ *with Reeb vector field* $\xi_{\mathbf{v}} \in \mathfrak{t}^+_{\mathbf{w}}$ *on* $M_{\mathbf{l},\mathbf{w}}$ *is a Kähler Bott orbifold* $\mathcal{K}_{\mathbf{m}_k,\mathbf{n}_k}$ *of the form* $(M_k(A(k)), \Delta^T_{\mathbf{m}}, \omega_k)$ *and* $A(k)$ *has the block form* $\left(\begin{array}{c|c} A(k-1) & 0 \\ \hline A(k)^1_k \cdots A(k)^{k-1}_k & 1 \end{array}\right)$*, where* $A(k)$ *is a matrix representative for the Bott manifold* M_k*. Moreover, the line orbibundle* L_k *defining the Bott tower* $(M_k(A(k)), \Delta^T_{\mathbf{m}_k}, \omega_k) \xrightarrow{\pi} (M_{k-1}(A(k-1)), \Delta^T_{\mathbf{m}_{k-1}}, \omega_{k-1})$ *satisfies* $c_1(L_k^{\Upsilon_{k-1}}) = \sum_{j=1}^{k-1} A(k)^j_k x_j = n_k[\omega_{k-1}]_I$*, and is determined by the Reeb vector field* $\xi_{\mathbf{v}_k}$ *where* $\mathbf{m}_k = m_k(v^0_k, v^\infty_k)$*,*

$$m_k = \frac{l^\infty_{k-1}}{s_k}, \qquad n_k = \frac{l^0_{k-1}}{s_k}(w^0_k v^\infty_k - w^\infty_k v^0_k), \qquad s_k = \gcd(l^\infty_{k-1}, |w^0_k v^\infty_k - w^\infty_k v^0_k|).$$

Note that $[\omega_i]_I =$ *is the appropriate integer (primitive) admissible Kähler class at each stage (vis a vis the previous theorem).*

Sasaki-Einstein

Let's go back to the setting of the first Theorem. Suppose (N, Δ_N) is positive Kähler-Einstein with Fano index $\mathcal{I}_N$, i.e. $c_1^{orb}(N, \Delta_N) = \mathcal{I}_N[\omega_N]$ and $l^0 = \frac{\mathcal{I}_N}{\gcd(w^0+w^\infty, \mathcal{I}_N)}$, $l^\infty = \frac{w^0+w^\infty}{\gcd(w^0+w^\infty, \mathcal{I}_N)}$, (ensures that $c_1(\mathcal{D})$ vanishes).

- Then $c_1^{orb}(S_n, \Delta_{\mathbf{m},N}) = \left(\frac{v^0+v^\infty}{s}\right)\gcd(l^0 w^0 v^\infty, s\Upsilon_N)[\omega_{n,\mathbf{m},N}]$ so
- the Fano index of $(S_n, \Delta_{\mathbf{m},N})$ (with $[\omega_{n,\mathbf{m},N}]$ the primitive KE class) is in turn $\mathcal{I}_{(S_n, \Delta_{\mathbf{m},N})} = \left(\frac{v^0+v^\infty}{s}\right)\gcd(l^0 w^0 v^\infty, s\Upsilon_N)$.
- and, coming from the admissible KE ODE, we always have an η-Einstein ray in $\mathfrak{t}_{\mathbf{w}}^+$ containing an Sasaki-Einstein structure (up to isotopy). This ray is the quasi-regular ray $\mathcal{S}_{\mathbf{v}}$ iff

$$\int_{-1}^{1} \left((1-b)-(1+b)\mathfrak{z}\right)\left((b+t)+(b-t)\mathfrak{z}\right)^{d_N} d\mathfrak{z} = 0, \qquad (1)$$

where $t = w^\infty/w^0$ $(0 < t < 1)$ and $b = v^\infty/v^0$.

Remark:

Guantlett, Martelli, Sparks, Waldram found all the smooth solutions, $Y^{p,q} = S^3 \star_{l,p} S^3_{\frac{p+q}{l},\frac{p-q}{l}}$, where $l = \gcd(p+q, p-q)$ and $N = \mathbb{CP}^1$.

Futaki, Ono, Wang showed a general existence result of (possibly irregular) Sasaki-Einstein structures in the Sasaki cone of any compact toric Sasaki manifold with positive basic first Chern class and vanishing first Chern class of the contact structure.

Iterating for SE:

- Building on this: start with a quasi-regular $Y^{p,q}$ and then iterate (in a non-trivial way) the join process again and again - solving (1) at each stage with evolving data $l^0, l^\infty, w^0, w^\infty$ as above - to produce some different kinds of explicit quasi-regular examples.
- At each step, to get **smooth** Sasaki-Einstein examples we need to satisfy (using Lemma 2 above) $\gcd(l^\infty \Upsilon, l^0 w^0 w^\infty) = 1$ where the order Υ increases as we go along.
- Note that the round Sasaki Einstein metric on S^3 is our a Stage 1 join ($S^1 \star_{1,1} S^3 = S^3$ where S^1 is then "Stage 0") and $Y^{p,q}$ is Stage 2.

Example

For all $t \in \mathbb{Z}^+$, $\mathcal{S}_3^t = Y^{780300t^2+65790t+1387,15(170t+7)(306t+13)} \star_{306\,t+13,4} S^3_{17,3}$ has a smooth quasi-regular Sasaki-Einstein structure. The quasi-regular quotient is a Stage 3 Bott orbifold given by the log pair $(M_3(a,b,c), \Delta_{\mathbf{m}})$ where $\Delta_{\mathbf{m}} = \sum_{i=1}^3 \left(\left(1 - \frac{1}{m_i^0}\right) D_{u_i^0} + \left(1 - \frac{1}{m_i^\infty}\right) D_{u_i^\infty} \right)$

$$
\begin{aligned}
a &= 6(255\,t + 11)(510\,t + 21)(1020\,t + 43) \\
b &= 204(255\,t + 11)(765\,t + 32)(1020\,t + 43)(306\,t + 13) \\
c &= 51(306\,t + 13) \\
\mathbf{m} &= (m_1^0, m_1^\infty, m_2^0, m_2^\infty, m_3^0, m_3^\infty) \\
(m_1^0, m_1^\infty) &= (1, 1) \\
(m_2^0, m_2^\infty) &= (1387 + 65790t + 780300t^2)(1020\,t + 43, 2(255\,t + 11)) \\
(m_3^0, m_3^\infty) &= 2(17, 9).
\end{aligned}
$$

Keep going...

We can iterate on the example above a few times to get:

Proposition

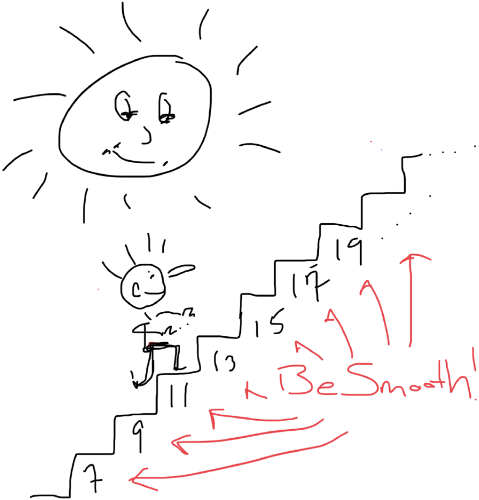
In dimensions 9 and 11 there exist countably infinite families of quasi-regular smooth Sasaki Einstein structures in the form of non-trivial iterated $S^3_{\mathbf{w}}$-joins $M^{2k+1}_{\mathbf{l},\mathbf{w}}$

Proof.

1. Calculate index (it is 13) and order (it is some polynomial in t) of the quasi-regular quotient $(M_3(a,b,c), \Delta_{\mathbf{m}})$ of $\mathcal{S}_3^t$ above.
2. Pick $(w_4^0, w_4^\infty) = (49, 13)$, $(l_3^0, l_3^\infty) = (13, 62)$ so $(v_4^0, v_4^\infty) = (49, 26)$ becomes an η-Einstein ray in $\mathcal{S}_3^t \star_{\mathbf{l}_3} S^3_{\mathbf{w}_4}$.
3. Adjust for Smoothness: Smoothness happens if $t = 7 \cdot 13 \cdot \hat{t}$ where $\hat{t} \in \mathbb{Z}^+$.
4. So now we have a one-parameter family of quasi-regular smooth, 9-dimensional Sasaki-Einstein structures $\mathcal{S}_4^{\hat{t}}$.
5. Calculate index (it is 150) and order (some polynomial in $\hat{t}$) of the quasi-regular quotient of $\mathcal{S}_4^{\hat{t}}$ above.
6. Pick $(w_5^0, w_5^\infty) = (25891157, 834997)$, $(l_4^0, l_4^\infty) = (25, 7 \cdot 13 \cdot 31 \cdot 1579)$ so $(v_5^0, v_5^\infty) = (3498805, 834997)$ becomes an η-Einstein ray in $\mathcal{S}_4^{\hat{t}} \star_{\mathbf{l}_4} S^3_{\mathbf{w}_5}$.
7. Adjust for Smoothness: Smoothness happens if $\hat{t} = 5^2 \cdot 29 \cdot 37 \cdot 28793 \cdot 699761 \tilde{t}$ where $\tilde{t} \in \mathbb{Z}^+$.
8. So now we have a one-parameter family of quasi-regular smooth, 11-dimensional Sasaki-Einstein structures $\mathcal{S}_5^{\tilde{t}}$.

□

9. Calculate index and order of the quasi-regular quotient of $\mathcal{S}_5^{\tilde{t}}$...



Conjecture:

There are iterated $S^3_{\mathbf{w}}$ joins (with no trivial iterations) admitting smooth Sasaki-Einstein structures in all odd dimensions.

Remark:

Due to the smoothness condition this is not just an easy induction...

What about just CSC iterations?

In theory this should be easier...but it is not...

Happy Retirement Professor Ahmed Zeriahi