Curvature of direct image of singular twisted relative canonical bundles

(joint with J. Cao and H. Guenancia)

Amazing Conference
June 2$^{\text{nd}}$, 2021

Mihai Paun
Universität Bayreuth, Germany

Notations

We will use the following notations:

- $p : \mathcal{X} \to \mathbb{D}$ proper, smooth Kähler family, $\mathcal{X}_t := p^{-1}(t)$.

Notations

We will use the following notations:

- $p : \mathcal{X} \to \mathbb{D}$ proper, smooth Kähler family, $\mathcal{X}_t := p^{-1}(t)$.
- $K_{\mathcal{X}/\mathbb{D}}$ is the relative canonical bundle of $\mathcal{X}$

Notations

We will use the following notations:

- $p : \mathcal{X} \to \mathbb{D}$ proper, smooth Kähler family, $\mathcal{X}_t := p^{-1}(t)$.
- $K_{\mathcal{X}/\mathbb{D}}$ is the relative canonical bundle of $\mathcal{X}$
- $E = \sum E_i$ divisor on $\mathcal{X}$ such that $E + \mathcal{X}_t$ snc for all $t \in \mathbb{D}$. Can choose $(z_1, \ldots, z_n, z_{n+1} = 1)$ coordinates on Ω such that $E \cap \Omega = (z_1 \ldots z_k = 0)$ and $p(z) = t$.

Notations

We will use the following notations:

- $p : \mathcal{X} \to \mathbb{D}$ proper, smooth Kähler family, $\mathcal{X}_t := p^{-1}(t)$.
- $K_{\mathcal{X}/\mathbb{D}}$ is the relative canonical bundle of $\mathcal{X}$
- $E = \sum E_i$ divisor on $\mathcal{X}$ such that $E + \mathcal{X}_t$ snc for all $t \in \mathbb{D}$. Can choose $(z_1, \ldots, z_n, z_{n+1} = 1)$ coordinates on Ω such that $E \cap \Omega = (z_1 ... z_k = 0)$ and $p(z) = t$.
- $(L, h_L) \to \mathcal{X}$ line bundle, $h_L = e^{-\varphi_L}$ such that modulo $\mathcal{C}^\infty$

Notations

We will use the following notations:

- $p : \mathcal{X} \to \mathbb{D}$ proper, smooth Kähler family, $\mathcal{X}_t := p^{-1}(t)$.
- $K_{\mathcal{X}/\mathbb{D}}$ is the relative canonical bundle of $\mathcal{X}$
- $E = \sum E_i$ divisor on $\mathcal{X}$ such that $E + \mathcal{X}_t$ snc for all $t \in \mathbb{D}$. Can choose $(z_1, \ldots, z_n, z_{n+1} = 1)$ coordinates on Ω such that $E \cap \Omega = (z_1 \ldots z_k = 0)$ and $p(z) = t$.
- $(L, h_L) \to \mathcal{X}$ line bundle, $h_L = e^{-\varphi_L}$ such that modulo $\mathcal{C}^\infty$

$$\varphi_L \simeq \sum_{i=1}^{k} a_i \log |z_i|^2 - \sum_{I \subset \{1,\ldots k\}} b_i \log \left(\phi_I - \log \prod_{i \in I} |z_i|^2 \right) \qquad a_i, b_I > 0.$$

Notations

We will use the following notations:

- $p : \mathcal{X} \to \mathbb{D}$ proper, smooth Kähler family, $\mathcal{X}_t := p^{-1}(t)$.
- $K_{\mathcal{X}/\mathbb{D}}$ is the relative canonical bundle of $\mathcal{X}$
- $E = \sum E_i$ divisor on $\mathcal{X}$ such that $E + \mathcal{X}_t$ snc for all $t \in \mathbb{D}$. Can choose $(z_1, \dots, z_n, z_{n+1} = 1)$ coordinates on Ω such that $E \cap \Omega = (z_1 ... z_k = 0)$ and $p(z) = t$.
- $(L, h_L) \to \mathcal{X}$ line bundle, $h_L = e^{-\varphi_L}$ such that modulo $\mathcal{C}^\infty$

$$\varphi_L \simeq \sum_{i=1}^{k} a_i \log |z_i|^2 - \sum_{I \subset \{1, \dots k\}} b_i \log \left(\phi_I - \log \prod_{i \in I} |z_i|^2 \right) \qquad a_i, b_I > 0.$$

- $\mathcal{I}(h_L)$ is the multiplier ideal sheaf of h_L.

Notations

We will use the following notations:

- $p : \mathcal{X} \to \mathbb{D}$ proper, smooth Kähler family, $\mathcal{X}_t := p^{-1}(t)$.
- $K_{\mathcal{X}/\mathbb{D}}$ is the relative canonical bundle of $\mathcal{X}$
- $E = \sum E_i$ divisor on $\mathcal{X}$ such that $E + \mathcal{X}_t$ snc for all $t \in \mathbb{D}$. Can choose $(z_1, \dots, z_n, z_{n+1} = 1)$ coordinates on Ω such that $E \cap \Omega = (z_1 \dots z_k = 0)$ and $p(z) = t$.
- $(L, h_L) \to \mathcal{X}$ line bundle, $h_L = e^{-\varphi_L}$ such that modulo $\mathcal{C}^\infty$

$$\varphi_L \simeq \sum_{i=1}^{k} a_i \log |z_i|^2 - \sum_{I \subset \{1,\dots k\}} b_i \log \Big(\phi_I - \log \prod_{i \in I} |z_i|^2 \Big) \qquad a_i, b_I > 0.$$

- $\mathcal{I}(h_L)$ is the multiplier ideal sheaf of h_L.
- $\mathcal{F} := p_\star\big((K_{\mathcal{X}/\mathbb{D}} + L) \otimes \mathcal{I}(h_L)\big)$. Note that we have

$$\mathcal{F}_t = H^0\Big(\mathcal{X}_t, (K_{\mathcal{X}_t} + L) \otimes \mathcal{I}(h_L|_{\mathcal{X}_t})\Big)$$

The main results, I

- Let $u, v \in \mathcal{C}^\infty(\mathbb{D}, \mathcal{F})$. Define $h_{\mathcal{F}}(u,v)_t := c_n \int_{\mathcal{X}_t} u \wedge \overline{v} e^{-\varphi_L}$

The main results, I

- Let $u, v \in \mathcal{C}^\infty(\mathbb{D}, \mathcal{F})$. Define $h_{\mathcal{F}}(u,v)_t := c_n \int_{\mathcal{X}_t} u \wedge \overline{v} e^{-\varphi_L}$
- Let $u \in \mathcal{C}^\infty(\mathbb{D}, \mathcal{F})$. A representative for u is a $(n, 0)$ form $\mathbf{u}$ on $\mathcal{X}$ with values in L such that

$$\mathbf{u}|_{\mathcal{X}_t} = u_t, \qquad \left.\frac{\bar{\partial}\mathbf{u}}{d\bar{t}}\right|_{\mathcal{X}_t} \in L^2$$

The main results, I

- Let $u, v \in \mathcal{C}^\infty(\mathbb{D}, \mathcal{F})$. Define $h_{\mathcal{F}}(u,v)_t := c_n \int_{\mathcal{X}_t} u \wedge \overline{v} e^{-\varphi_L}$
- Let $u \in \mathcal{C}^\infty(\mathbb{D}, \mathcal{F})$. A representative for u is a $(n, 0)$ form $\mathbf{u}$ on $\mathcal{X}$ with values in L such that

$$\mathbf{u}|_{\mathcal{X}_t} = u_t, \qquad \left.\frac{\bar\partial \mathbf{u}}{d\bar t}\right|_{\mathcal{X}_t} \in L^2$$

- Let $D_{\mathcal{F}} = D'_{\mathcal{F}} + \bar\partial$ be the induced Chern connection.

The main results, I

- Let $u, v \in \mathcal{C}^\infty(\mathbb{D}, \mathcal{F})$. Define $h_{\mathcal{F}}(u,v)_t := c_n \int_{\mathcal{X}_t} u \wedge \overline{v} e^{-\varphi_L}$
- Let $u \in \mathcal{C}^\infty(\mathbb{D}, \mathcal{F})$. A representative for u is a $(n,0)$ form $\mathbf{u}$ on $\mathcal{X}$ with values in L such that

$$\mathbf{u}|_{\mathcal{X}_t} = u_t, \qquad \frac{\bar\partial \mathbf{u}}{d\bar t}\Big|_{\mathcal{X}_t} \in L^2$$

- Let $D_{\mathcal{F}} = D'_{\mathcal{F}} + \bar\partial$ be the induced Chern connection.

Theorem 1 [CGP]

Let $p : \mathcal{X} \to \mathbb{D}$ and $(L, h_L) \to \mathcal{X}$ as above (with $b_I = 0$) and let $u \in H^0(\mathbb{D}, \mathcal{F})$. We assume that

$$\sqrt{-1}\Theta(L, h_L) \geq 0, \qquad D'_{\mathcal{F}} u = 0.$$

The main results, I

- Let $u, v \in \mathcal{C}^\infty(\mathbb{D}, \mathcal{F})$. Define $h_{\mathcal{F}}(u,v)_t := c_n \int_{\mathcal{X}_t} u \wedge \overline{v} e^{-\varphi_L}$
- Let $u \in \mathcal{C}^\infty(\mathbb{D}, \mathcal{F})$. A representative for u is a $(n,0)$ form $\mathbf{u}$ on $\mathcal{X}$ with values in L such that

$$\mathbf{u}|_{\mathcal{X}_t} = u_t, \qquad \frac{\bar{\partial}\mathbf{u}}{d\bar{t}}\bigg|_{\mathcal{X}_t} \in L^2$$

- Let $D_{\mathcal{F}} = D'_{\mathcal{F}} + \bar{\partial}$ be the induced Chern connection.

Theorem 1 [CGP]

Let $p : \mathcal{X} \to \mathbb{D}$ and $(L, h_L) \to \mathcal{X}$ as above (with $b_I = 0$) and let $u \in H^0(\mathbb{D}, \mathcal{F})$. We assume that

$$\sqrt{-1}\Theta(L, h_L) \geq 0, \qquad D'_{\mathcal{F}} u = 0.$$

Then there exists a continuous L^2 representative $\mathbf{u}$ of u defined on $p : \mathcal{X}^\star \setminus E \to \mathbb{D}^\star$ such that

The main results, I

- Let $u, v \in \mathcal{C}^\infty(\mathbb{D}, \mathcal{F})$. Define $h_{\mathcal{F}}(u,v)_t := c_n \int_{\mathcal{X}_t} u \wedge \overline{v} e^{-\varphi_L}$
- Let $u \in \mathcal{C}^\infty(\mathbb{D}, \mathcal{F})$. A representative for u is a $(n, 0)$ form $\mathbf{u}$ on $\mathcal{X}$ with values in L such that

$$\mathbf{u}|_{\mathcal{X}_t} = u_t, \qquad \frac{\bar{\partial}\mathbf{u}}{d\bar{t}}\Big|_{\mathcal{X}_t} \in L^2$$

- Let $D_{\mathcal{F}} = D'_{\mathcal{F}} + \bar{\partial}$ be the induced Chern connection.

Theorem 1 [CGP]

Let $p : \mathcal{X} \to \mathbb{D}$ and $(L, h_L) \to \mathcal{X}$ as above (with $b_I = 0$) and let $u \in H^0(\mathbb{D}, \mathcal{F})$. We assume that

$$\sqrt{-1}\Theta(L, h_L) \geq 0, \qquad D'_{\mathcal{F}} u = 0.$$

Then there exists a continuous L^2 representative $\mathbf{u}$ of u defined on $p : \mathcal{X}^\star \setminus E \to \mathbb{D}^\star$ such that

$$\frac{\bar{\partial}\mathbf{u}}{dt}\Big|_{X_t \setminus E} = 0, \qquad D'\mathbf{u} = 0, \qquad \Theta_{h_L}(L) \wedge \mathbf{u} = 0.$$

on $\mathcal{X}^\star \setminus E$, $t \in D^\star$.

The main results, II

- We will also discuss the following.

The main results, II

- We will also discuss the following.

Theorem 2 [CGP]

Let $p : \mathcal{X} \to \mathbb{D}$ be a smooth projective fibration and let $(L, h_L) \to \mathcal{X}$ be a line bundle as above, together with

The main results, II

- We will also discuss the following.

Theorem 2 [CGP]

Let $p : \mathcal{X} \to \mathbb{D}$ be a smooth projective fibration and let $(L, h_L) \to \mathcal{X}$ be a line bundle as above, together with

1. $\sqrt{-1}\Theta_{h_L}(L) \geq 0$

The main results, II

- We will also discuss the following.

Theorem 2 [CGP]

Let $p : \mathcal{X} \to \mathbb{D}$ be a smooth projective fibration and let $(L, h_L) \to \mathcal{X}$ be a line bundle as above, together with

1. $\sqrt{-1}\Theta_{h_L}(L) \geq 0$
2. For any $t \in \mathbb{D}$, the absolutely continuous part $\omega_L := \sqrt{-1}\Theta_{h_L}(L)_{\mathrm{ac}}$ satisfies $\int_{X_t} \omega_L^n > 0$.

The main results, II

- We will also discuss the following.

Theorem 2 [CGP]

Let $p : \mathcal{X} \to \mathbb{D}$ be a smooth projective fibration and let $(L, h_L) \to \mathcal{X}$ be a line bundle as above, together with

1. $\sqrt{-1}\Theta_{h_L}(L) \geq 0$
2. For any $t \in \mathbb{D}$, the absolutely continuous part $\omega_L := \sqrt{-1}\Theta_{h_L}(L)_{\mathrm{ac}}$ satisfies $\int_{X_t} \omega_L^n > 0$.

Then there exists $D^\star \subset D$ such that $\forall t \in \mathbb{D}^\star$ and for any $u \in H^0(\mathbb{D}, \mathcal{F})$

$$\langle \sqrt{-1}\Theta_{h_{\mathcal{F}}}(\mathcal{F})u, u\rangle_t \geq c_n \int_{X_t} c(\omega_L) u \wedge \overline{u} e^{-\phi_L}$$

The main results, II

- We will also discuss the following.

Theorem 2 [CGP]

Let $p : \mathcal{X} \to \mathbb{D}$ be a smooth projective fibration and let $(L, h_L) \to \mathcal{X}$ be a line bundle as above, together with

1. $\sqrt{-1}\Theta_{h_L}(L) \geq 0$
2. For any $t \in \mathbb{D}$, the absolutely continuous part $\omega_L := \sqrt{-1}\Theta_{h_L}(L)_{\mathrm{ac}}$ satisfies $\int_{X_t} \omega_L^n > 0$.

Then there exists $D^\star \subset D$ such that $\forall t \in \mathbb{D}^\star$ and for any $u \in H^0(\mathbb{D}, \mathcal{F})$

$$\langle \sqrt{-1}\Theta_{h_{\mathcal{F}}}(\mathcal{F})u, u \rangle_t \geq c_n \int_{X_t} c(\omega_L) u \wedge \overline{u} e^{-\phi_L}$$

- $c(\omega_L) := \dfrac{\omega_L^{n+1}}{\omega_L^n \wedge idt \wedge d\overline{t}}$ the geodesic curvature associated to ω_L (defined by approximation in the degenerate case).

Motivation

- The following important problem is (still...) open.

Motivation

- The following important problem is (still...) open.

Conjecture [Iitaka]

Let (X, B) be a projective manifold together with an effective $\mathbb{Q}$-divisor B such that $\mathcal{I}(B) = \mathcal{O}_X$. Then

$$\kappa(X, B) \geq \kappa(X_t, B_t) + \kappa(Y)$$

$\kappa =$ Kodaira dimension (growth order of the space of pluricanonical sections).

Motivation

- The following important problem is (still...) open.

Conjecture [Iitaka]

Let (X, B) be a projective manifold together with an effective $\mathbb{Q}$-divisor B such that $\mathcal{I}(B) = \mathcal{O}_X$. Then

$$\kappa(X, B) \geq \kappa(X_t, B_t) + \kappa(Y)$$

$\kappa =$ Kodaira dimension (growth order of the space of pluricanonical sections).

A few remarks:

- In all the known particular cases the sheaf $\mathcal{F}_m := p_\star\big(m(K_{X/Y} + B)\big)$ plays a crucial role

Motivation

- The following important problem is (still...) open.

Conjecture [Iitaka]

Let (X, B) be a projective manifold together with an effective $\mathbb{Q}$-divisor B such that $\mathcal{I}(B) = \mathcal{O}_X$. Then

$$\kappa(X, B) \geq \kappa(X_t, B_t) + \kappa(Y)$$

$\kappa =$ Kodaira dimension (growth order of the space of pluricanonical sections).

A few remarks:

- In all the known particular cases the sheaf $\mathcal{F}_m := p_\star\big(m(K_{X/Y} + B)\big)$ plays a crucial role
- One can construct a natural, positively curved metric h_m on $\mathcal{F}_m$.

Motivation

- The following important problem is (still...) open.

Conjecture [Iitaka]

Let (X, B) be a projective manifold together with an effective $\mathbb{Q}$-divisor B such that $\mathcal{I}(B) = \mathcal{O}_X$. Then

$$\kappa(X, B) \geq \kappa(X_t, B_t) + \kappa(Y)$$

$\kappa =$ Kodaira dimension (growth order of the space of pluricanonical sections).

A few remarks:

- In all the known particular cases the sheaf $\mathcal{F}_m := p_\star\big(m(K_{X/Y} + B)\big)$ plays a crucial role
- One can construct a natural, positively curved metric h_m on $\mathcal{F}_m$.
- Consider $\mathcal{L}_m := \det(\mathcal{F}_m)$; conjecture known if $\mathcal{L}_m$ big or $c_1(\mathcal{L}_m) = 0$.

Motivation

- The following important problem is (still...) open.

Conjecture [Iitaka]

Let (X, B) be a projective manifold together with an effective $\mathbb{Q}$-divisor B such that $\mathcal{I}(B) = \mathcal{O}_X$. Then

$$\kappa(X, B) \geq \kappa(X_t, B_t) + \kappa(Y)$$

$\kappa =$ Kodaira dimension (growth order of the space of pluricanonical sections).

A few remarks:

- In all the known particular cases the sheaf $\mathcal{F}_m := p_\star\big(m(K_{X/Y} + B)\big)$ plays a crucial role
- One can construct a natural, positively curved metric h_m on $\mathcal{F}_m$.
- Consider $\mathcal{L}_m := \det(\mathcal{F}_m)$; conjecture known if $\mathcal{L}_m$ big or $c_1(\mathcal{L}_m) = 0$.
- Theorem 1: attempt to understand better the intermediate case.

Motivation

- The following important problem is (still...) open.

Conjecture [Iitaka]

Let (X, B) be a projective manifold together with an effective $\mathbb{Q}$-divisor B such that $\mathcal{I}(B) = \mathcal{O}_X$. Then

$$\kappa(X, B) \geq \kappa(X_t, B_t) + \kappa(Y)$$

$\kappa =$ Kodaira dimension (growth order of the space of pluricanonical sections).

A few remarks:

- In all the known particular cases the sheaf $\mathcal{F}_m := p_\star\big(m(K_{X/Y} + B)\big)$ plays a crucial role
- One can construct a natural, positively curved metric h_m on $\mathcal{F}_m$.
- Consider $\mathcal{L}_m := \det(\mathcal{F}_m)$; conjecture known if $\mathcal{L}_m$ big or $c_1(\mathcal{L}_m) = 0$.
- Theorem 1: attempt to understand better the intermediate case.

- We discuss next the main ingredients in the proof.

The non-singular case: Berndtsson's approach

• In his work concerning the positivity properties of $(\mathcal{F}, h_{\mathcal{F}})$, Berndtsson is using the following technique.

The non-singular case: Berndtsson's approach

• In his work concerning the positivity properties of $(\mathcal{F}, h_{\mathcal{F}})$, Berndtsson is using the following technique.

- Let ω be a Kähler metric on $\mathcal{X}$. Consider $u \in H^0(\mathbb{D}, \mathcal{F})$ and $\mathbf{u} =$ any representative of u.

The non-singular case: Berndtsson's approach

• In his work concerning the positivity properties of $(\mathcal{F}, h_{\mathcal{F}})$, Berndtsson is using the following technique.

- Let ω be a Kähler metric on $\mathcal{X}$. Consider $u \in H^0(\mathbb{D}, \mathcal{F})$ and $\mathbf{u} =$ any representative of u.
- In case of a non-singular metric h_L we have the following formula

The non-singular case: Berndtsson's approach

• In his work concerning the positivity properties of $(\mathcal{F}, h_{\mathcal{F}})$, Berndtsson is using the following technique.

- Let ω be a Kähler metric on $\mathcal{X}$. Consider $u \in H^0(\mathbb{D}, \mathcal{F})$ and $\mathbf{u}$ = any representative of u.
- In case of a non-singular metric h_L we have the following formula

$$\partial\bar{\partial}\|u\|^2_{h_{\mathcal{F}}} = c_n\Big[- p_\star(\Theta_{h_L}(L)\wedge \mathbf{u}\wedge \bar{\mathbf{u}}e^{-\varphi_L}) + (-1)^n p_\star(D'\mathbf{u}\wedge \overline{D'\mathbf{u}}e^{-\varphi_L}) + (-1)^n p_\star(\bar{\partial}\mathbf{u}\wedge \overline{\bar{\partial}\mathbf{u}}\, e^{-\varphi_L})\Big]$$

The non-singular case: Berndtsson's approach

- In his work concerning the positivity properties of $(\mathcal{F}, h_{\mathcal{F}})$, Berndtsson is using the following technique.
  - Let ω be a Kähler metric on $\mathcal{X}$. Consider $u \in H^0(\mathbb{D}, \mathcal{F})$ and $\mathbf{u}$ = any representative of u.
  - In case of a non-singular metric h_L we have the following formula

$$\partial\bar{\partial}\|u\|^2_{h_{\mathcal{F}}} = c_n\Big[- p_\star(\Theta_{h_L}(L) \wedge \mathbf{u} \wedge \bar{\mathbf{u}} e^{-\varphi_L}) + (-1)^n p_\star(D'\mathbf{u} \wedge \overline{D'\mathbf{u}} e^{-\varphi_L}) + (-1)^n p_\star(\bar{\partial}\mathbf{u} \wedge \overline{\bar{\partial}\mathbf{u}}\, e^{-\varphi_L})\Big]$$

- The non-singular version Theorem 1 follows:
  - We have u such that $D'_{\mathcal{F}} u = 0$.

The non-singular case: Berndtsson's approach

- In his work concerning the positivity properties of $(\mathcal{F}, h_{\mathcal{F}})$, Berndtsson is using the following technique.
  - Let ω be a Kähler metric on $\mathcal{X}$. Consider $u \in H^0(\mathbb{D}, \mathcal{F})$ and $\mathbf{u} =$ any representative of u.
  - In case of a non-singular metric h_L we have the following formula

$$\partial\bar\partial\|u\|^2_{h_{\mathcal{F}}} = c_n\Big[-p_\star(\Theta_{h_L}(L)\wedge\mathbf{u}\wedge\bar{\mathbf{u}}e^{-\varphi_L}) + (-1)^n p_\star(D'\mathbf{u}\wedge\overline{D'\mathbf{u}}e^{-\varphi_L}) + (-1)^n p_\star(\bar\partial\mathbf{u}\wedge\overline{\bar\partial\mathbf{u}}\,e^{-\varphi_L})\Big]$$

- The non-singular version Theorem 1 follows:
  - We have u such that $D'_{\mathcal{F}}u = 0$.
  - Hodge theory shows that $\exists\mathbf{u}$ such that

The non-singular case: Berndtsson's approach

- In his work concerning the positivity properties of $(\mathcal{F}, h_{\mathcal{F}})$, Berndtsson is using the following technique.
  - Let ω be a Kähler metric on $\mathcal{X}$. Consider $u \in H^0(\mathbb{D}, \mathcal{F})$ and $\mathbf{u} =$ any representative of u.
  - In case of a non-singular metric h_L we have the following formula

$$\partial\bar{\partial}\|u\|^2_{h_{\mathcal{F}}} = c_n\Big[-p_\star(\Theta_{h_L}(L)\wedge\mathbf{u}\wedge\bar{\mathbf{u}}e^{-\varphi_L}) + (-1)^n p_\star(D'\mathbf{u}\wedge\overline{D'\mathbf{u}}e^{-\varphi_L}) + (-1)^n p_\star(\bar{\partial}\mathbf{u}\wedge\overline{\bar{\partial}\mathbf{u}}\,e^{-\varphi_L})\Big]$$

- The non-singular version Theorem 1 follows:
  - We have u such that $D'_{\mathcal{F}}u = 0$.
  - Hodge theory shows that $\exists\mathbf{u}$ such that

$$\bar{\partial}\mathbf{u} = dt\wedge\eta, \qquad D'\mathbf{u} = dt\wedge\mu$$

The non-singular case: Berndtsson's approach

- In his work concerning the positivity properties of $(\mathcal{F}, h_{\mathcal{F}})$, Berndtsson is using the following technique.
  - Let ω be a Kähler metric on $\mathcal{X}$. Consider $u \in H^0(\mathbb{D}, \mathcal{F})$ and $\mathbf{u}$ = any representative of u.
  - In case of a non-singular metric h_L we have the following formula

$$\partial\bar{\partial}\|u\|^2_{h_{\mathcal{F}}} = c_n\Big[-p_\star(\Theta_{h_L}(L)\wedge \mathbf{u}\wedge \bar{\mathbf{u}}e^{-\varphi_L}) + (-1)^n p_\star(D'\mathbf{u}\wedge \overline{D'\mathbf{u}}e^{-\varphi_L}) + (-1)^n p_\star(\bar{\partial}\mathbf{u}\wedge \overline{\bar{\partial}\mathbf{u}}\, e^{-\varphi_L})\Big]$$

- The non-singular version Theorem 1 follows:
  - We have u such that $D'_{\mathcal{F}}u = 0$.
  - Hodge theory shows that $\exists\mathbf{u}$ such that

$$\bar{\partial}\mathbf{u} = dt\wedge\eta, \qquad D'\mathbf{u} = dt\wedge\mu$$

  on each $\mathcal{X}_t$, where $\eta\wedge\omega|_{\mathcal{X}_t} = 0$ and $\mu|_{\mathcal{X}_t} = 0$. This is achieved by solving a fiber-wise $\bar{\partial}^\star$ equation.

The non-singular case: Berndtsson's approach

- In his work concerning the positivity properties of $(\mathcal{F}, h_{\mathcal{F}})$, Berndtsson is using the following technique.
  - Let ω be a Kähler metric on $\mathcal{X}$. Consider $u \in H^0(\mathbb{D}, \mathcal{F})$ and $\mathbf{u}$ = any representative of u.
  - In case of a non-singular metric h_L we have the following formula

$$\partial\bar\partial\|u\|^2_{h_{\mathcal{F}}} = c_n\Big[-p_\star(\Theta_{h_L}(L)\wedge\mathbf{u}\wedge\bar{\mathbf{u}}e^{-\varphi_L}) + (-1)^n p_\star(D'\mathbf{u}\wedge\overline{D'\mathbf{u}}e^{-\varphi_L}) + (-1)^n p_\star(\bar\partial\mathbf{u}\wedge\overline{\bar\partial\mathbf{u}}\,e^{-\varphi_L})\Big]$$

- The non-singular version Theorem 1 follows:
  - We have u such that $D'_{\mathcal{F}}u = 0$.
  - Hodge theory shows that $\exists\mathbf{u}$ such that

$$\bar\partial\mathbf{u} = dt\wedge\eta, \qquad D'\mathbf{u} = dt\wedge\mu$$

    on each $\mathcal{X}_t$, where $\eta\wedge\omega|_{\mathcal{X}_t} = 0$ and $\mu|_{\mathcal{X}_t} = 0$. This is achieved by solving a fiber-wise $\bar\partial^\star$ equation.
  - The LHS of formula is zero; Theorem 1 follows by using the representative $\mathbf{u}$ above.

The general case: construction of representatives

- Back to the general setting (i.e. h_L as in a few clicks above)

The general case: construction of representatives

- Back to the general setting (i.e. h_L as in a few clicks above)
- We use a metric on $\mathcal{X}$ with Poincaré singularities along E

The general case: construction of representatives

- Back to the general setting (i.e. h_L as in a few clicks above)
- We use a metric on $\mathcal{X}$ with Poincaré singularities along E

$$\omega_E := \omega + \sqrt{-1}\partial\bar{\partial}\Big[- \sum_{i=1}^{N} \log\log\frac{1}{|s_i|^2}\Big]$$

The general case: construction of representatives

- Back to the general setting (i.e. h_L as in a few clicks above)
- We use a metric on $\mathcal{X}$ with Poincaré singularities along E

$$\omega_E := \omega + \sqrt{-1}\partial\bar{\partial}\Big[-\sum_{i=1}^{N} \log\log \frac{1}{|s_i|^2}\Big]$$

- Consider local co-ordinates $(z_1, \ldots, z_n, z_{n+1} = t)$ such that $p(z) = t$

$$\omega_E|_\Omega = g_{t\bar{t}}\, idt \wedge d\bar{t} + \sum_\alpha g_{\alpha\bar{t}}\, idz_\alpha \wedge d\bar{t} + \sum_\alpha g_{t\bar{\alpha}}\, idt \wedge d\bar{z}_\alpha + \sum_{\alpha,\beta} g_{\alpha\bar{\beta}}\, idz_\alpha \wedge d\bar{z}_\beta$$

The general case: construction of representatives

- Back to the general setting (i.e. h_L as in a few clicks above)
- We use a metric on $\mathcal{X}$ with Poincaré singularities along E

$$\omega_E := \omega + \sqrt{-1}\partial\bar{\partial}\Big[- \sum_{i=1}^{N} \log\log \frac{1}{|s_i|^2} \Big]$$

- Consider local co-ordinates $(z_1, \ldots, z_n, z_{n+1} = t)$ such that $p(z) = t$

$$\omega_E|_\Omega = g_{t\bar{t}}\, idt \wedge d\bar{t} + \sum_\alpha g_{\alpha\bar{t}}\, idz_\alpha \wedge d\bar{t} + \sum_\alpha g_{t\bar{\alpha}}\, idt \wedge d\bar{z}_\alpha + \sum_{\alpha,\beta} g_{\alpha\bar{\beta}}\, idz_\alpha \wedge d\bar{z}_\beta$$

- The horizontal lifting of $\frac{\partial}{\partial t}$ given by

$$V := \frac{\partial}{\partial t} - \sum_{\alpha,\beta} g^{\bar{\beta}\alpha} g_{t\bar{\beta}} \frac{\partial}{\partial z_\alpha}$$

The general case: construction of representatives

- Back to the general setting (i.e. h_L as in a few clicks above)
- We use a metric on $\mathcal{X}$ with Poincaré singularities along E

$$\omega_E := \omega + \sqrt{-1}\partial\bar{\partial}\Big[- \sum_{i=1}^{N} \log\log \frac{1}{|s_i|^2}\Big]$$

- Consider local co-ordinates $(z_1, \dots, z_n, z_{n+1} = t)$ such that $p(z) = t$

$$\omega_E|_\Omega = g_{t\bar{t}}\, idt \wedge d\bar{t} + \sum_\alpha g_{\alpha\bar{t}}\, idz_\alpha \wedge d\bar{t} + \sum_\alpha g_{t\bar{\alpha}}\, idt \wedge d\bar{z}_\alpha + \sum_{\alpha,\beta} g_{\alpha\bar{\beta}}\, idz_\alpha \wedge d\bar{z}_\beta$$

- The horizontal lifting of $\dfrac{\partial}{\partial t}$ given by

$$V := \frac{\partial}{\partial t} - \sum_{\alpha,\beta} g^{\bar{\beta}\alpha} g_{t\bar{\beta}} \frac{\partial}{\partial z_\alpha}$$

- Let u be a section of $\mathcal{F}$. We define

$$\mathbf{u} := V \rfloor (dt \wedge U_0)$$

where U_0 is an arbitrary representative of u.

Properties of $\mathbf{u}$ and a general curvature formula

- We define $\bar{\partial}\mathbf{u} = dt \wedge \eta$ and $D'\mathbf{u} = dt \wedge \mu$.

Properties of $\mathbf{u}$ and a general curvature formula

- We define $\bar\partial \mathbf{u} = dt \wedge \eta$ and $D'\mathbf{u} = dt \wedge \mu$.
  - The forms $\eta|_{\mathcal{X}_t}, \mu|_{\mathcal{X}_t}$ and $\mu|_{\mathcal{X}_t}$ are L^2.

Properties of $\mathbf{u}$ and a general curvature formula

- We define $\bar{\partial}\mathbf{u} = dt \wedge \eta$ and $D'\mathbf{u} = dt \wedge \mu$.
  - The forms $\eta|_{\mathcal{X}_t}, \mu|_{\mathcal{X}_t}$ and $\mu|_{\mathcal{X}_t}$ are L^2.
  - The forms $\mathbf{u}, \eta, \mu$ and $\bar{\partial}\mu$ are also in $L^2(\mathcal{X})$

Properties of $\mathbf{u}$ and a general curvature formula

- We define $\bar\partial \mathbf{u} = dt \wedge \eta$ and $D'\mathbf{u} = dt \wedge \mu$.
  - The forms $\eta|_{\mathcal{X}_t}, \mu|_{\mathcal{X}_t}$ and $\mu|_{\mathcal{X}_t}$ are L^2.
  - The forms $\mathbf{u}, \eta, \mu$ and $\bar\partial\mu$ are also in $L^2(\mathcal{X})$
  - We also have $\mathbf{u} \wedge \omega_E|_\Omega = a(z,t) g_{t\bar t}\, dt \wedge d\bar t \wedge dz_1 \wedge \ldots \wedge dz_n$. Thus

$$\left.\frac{\mathbf{u} \wedge \omega_E}{dt}\right|_{\mathcal{X}_t} = 0.$$

Properties of $\mathbf{u}$ and a general curvature formula

- We define $\bar{\partial}\mathbf{u} = dt \wedge \eta$ and $D'\mathbf{u} = dt \wedge \mu$.
  - The forms $\eta|_{\mathcal{X}_t}, \mu|_{\mathcal{X}_t}$ and $\mu|_{\mathcal{X}_t}$ are L^2.
  - The forms $\mathbf{u}, \eta, \mu$ and $\bar{\partial}\mu$ are also in $L^2(\mathcal{X})$
  - We also have $\mathbf{u} \wedge \omega_E|_\Omega = a(z,t) g_{t\bar{t}}\, dt \wedge d\bar{t} \wedge dz_1 \wedge \ldots \wedge dz_n$. Thus
$$\left.\frac{\mathbf{u} \wedge \omega_E}{dt}\right|_{\mathcal{X}_t} = 0.$$
  - It follows that $\eta \wedge \omega_E|_{\mathcal{X}_t} = 0$.

Properties of $\mathbf{u}$ and a general curvature formula

- We define $\bar\partial \mathbf{u} = dt \wedge \eta$ and $D'\mathbf{u} = dt \wedge \mu$.
  - The forms $\eta|_{\mathcal{X}_t}, \mu|_{\mathcal{X}_t}$ and $\mu|_{\mathcal{X}_t}$ are L^2.
  - The forms $\mathbf{u}, \eta, \mu$ and $\bar\partial\mu$ are also in $L^2(\mathcal{X})$
  - We also have $\mathbf{u} \wedge \omega_E|_\Omega = a(z,t) g_{t\bar t}\, dt \wedge d\bar t \wedge dz_1 \wedge \ldots \wedge dz_n$. Thus

$$\left.\frac{\mathbf{u} \wedge \omega_E}{dt}\right|_{\mathcal{X}_t} = 0.$$

  - It follows that $\eta \wedge \omega_E|_{\mathcal{X}_t} = 0$.

Proposition

Let $\mathbf{u}$ be a continuous representative of u as above. Then

Properties of $\mathbf{u}$ and a general curvature formula

- We define $\bar\partial \mathbf{u} = dt \wedge \eta$ and $D'\mathbf{u} = dt \wedge \mu$.
  - The forms $\eta|_{\mathcal{X}_t}, \mu|_{\mathcal{X}_t}$ and $\mu|_{\mathcal{X}_t}$ are L^2.
  - The forms $\mathbf{u}, \eta, \mu$ and $\bar\partial \mu$ are also in $L^2(\mathcal{X})$
  - We also have $\mathbf{u} \wedge \omega_E|_\Omega = a(z,t) g_{t\bar t}\, dt \wedge d\bar t \wedge dz_1 \wedge \ldots \wedge dz_n$. Thus

$$\left.\frac{\mathbf{u} \wedge \omega_E}{dt}\right|_{\mathcal{X}_t} = 0.$$

  - It follows that $\eta \wedge \omega_E|_{\mathcal{X}_t} = 0$.

Proposition

Let $\mathbf{u}$ be a continuous representative of u as above. Then

$$\begin{aligned}\partial\bar\partial \|u\|^2_{h_{\mathcal{F}}} = c_n \Big[& - p_\star(\Theta_{h_L}(L)_{\mathrm{ac}} \wedge \mathbf{u} \wedge \bar{\mathbf{u}} e^{-\phi_L}) + (-1)^n p_\star(D'\mathbf{u} \wedge \overline{D'\mathbf{u}} e^{-\phi_L}) \\ & + (-1)^n p_\star(\bar\partial \mathbf{u} \wedge \overline{\bar\partial \mathbf{u}}\, e^{-\phi_L}) \Big]\end{aligned}$$

Here $\Theta_{h_L}(L)_{\mathrm{ac}}$ is the absolutely continuous part of the current $\Theta_{h_L}(L)$.

Hodge decomposition

• To argue as in the smooth case we need a version of the Hodge decomposition in the following setting. Let X be a compact Kähler manifold and let $E = E_1 + \cdots + E_k$ be a snc divisor.

Hodge decomposition

• To argue as in the smooth case we need a version of the Hodge decomposition in the following setting. Let X be a compact Kähler manifold and let $E = E_1 + \cdots + E_k$ be a snc divisor.

• We consider ω_E a metric with Poincaré-type singularities along E and (L, h_L) as above.

Hodge decomposition

• To argue as in the smooth case we need a version of the Hodge decomposition in the following setting. Let X be a compact Kähler manifold and let $E = E_1 + \cdots + E_k$ be a snc divisor.

• We consider ω_E a metric with Poincaré-type singularities along E and (L, h_L) as above.

Theorem 3 [CP]

We have the following equality for (X, ω_E) and (L, h_L).

Hodge decomposition

• To argue as in the smooth case we need a version of the Hodge decomposition in the following setting. Let X be a compact Kähler manifold and let $E = E_1 + \cdots + E_k$ be a snc divisor.

• We consider ω_E a metric with Poincaré-type singularities along E and (L, h_L) as above.

Theorem 3 [CP]

We have the following equality for (X, ω_E) and (L, h_L).

$$L^2_{n,1}(X_0, L) = \mathcal{H}_{n,1}(X_0, L) \oplus \mathrm{Im}\overline{\partial} \oplus \mathrm{Im}\overline{\partial}^\star$$

where $X_0 := X \setminus Y$.

Hodge decomposition

• To argue as in the smooth case we need a version of the Hodge decomposition in the following setting. Let X be a compact Kähler manifold and let $E = E_1 + \cdots + E_k$ be a snc divisor.

• We consider ω_E a metric with Poincaré-type singularities along E and (L, h_L) as above.

Theorem 3 [CP]

We have the following equality for (X, ω_E) and (L, h_L).

$$L^2_{n,1}(X_0, L) = \mathcal{H}_{n,1}(X_0, L) \oplus \mathrm{Im}\overline{\partial} \oplus \mathrm{Im}\overline{\partial}^\star$$

where $X_0 := X \setminus Y$.

• This can be seen as part of L^2-Hodge theory (cf. work by A. Fujiki, S. Zucker, Pardon-Stein and more recently H. Auvray, P. Naumann...).

Hodge decomposition

• To argue as in the smooth case we need a version of the Hodge decomposition in the following setting. Let X be a compact Kähler manifold and let $E = E_1 + \cdots + E_k$ be a snc divisor.

• We consider ω_E a metric with Poincaré-type singularities along E and (L, h_L) as above.

Theorem 3 [CP]

We have the following equality for (X, ω_E) and (L, h_L).

$$L^2_{n,1}(X_0, L) = \mathcal{H}_{n,1}(X_0, L) \oplus \mathrm{Im}\overline{\partial} \oplus \mathrm{Im}\overline{\partial}^{\star}$$

where $X_0 := X \setminus Y$.

• This can be seen as part of L^2-Hodge theory (cf. work by A. Fujiki, S. Zucker, Pardon-Stein and more recently H. Auvray, P. Naumann...).

• The proof based on the fact that (X_0, ω_E) complete, together with the following a-priori estimate.

Hodge decomposition, II

- Let $A := [\sqrt{-1}\Theta_{h_L}(L), \Lambda_{\omega_E}]$ be the usual curvature operator. We consider

$$H^{(p)} := \{v \in H^0(X^\circ, \Omega^p_{X^\circ} \otimes L) \cap L^2; \int_{X^\circ} \langle A \star v, \star v \rangle dV_{\omega_E} = 0\}.$$

Hodge decomposition, II

- Let $A := [\sqrt{-1}\Theta_{h_L}(L), \Lambda_{\omega_E}]$ be the usual curvature operator. We consider

$$H^{(p)} := \{v \in H^0(X^\circ, \Omega^p_{X^\circ} \otimes L) \cap L^2; \int_{X^\circ} \langle A \star v, \star v \rangle dV_{\omega_E} = 0\}.$$

- The following is an important ingredient in establishing the Hodge decomposition.

Theorem 4 (Poincaré inequality)

Let $p \leq n$ be a positive integer. There exists a positive constant $C > 0$ such that

$$\int_{X_0} |u|^2_{\omega_E} e^{-\varphi_L} dV \leq C \left(\int_{X_0} |\overline{\partial} u|^2_{\omega_E} e^{-\varphi_L} dV_{\omega_E} + \int_{X_0} \langle A \star u, \star u \rangle dV_{\omega_E} \right)$$

for any L-valued form u of type $(p, 0)$ which belongs to the domain of $\bar{\partial}$ and which is orthogonal to the space $H^{(p)}$.

Hodge decomposition, II

- Let $A := [\sqrt{-1}\Theta_{h_L}(L), \Lambda_{\omega_E}]$ be the usual curvature operator. We consider

$$H^{(p)} := \{v \in H^0(X^\circ, \Omega^p_{X^\circ} \otimes L) \cap L^2;\ \int_{X^\circ} \langle A \star v, \star v \rangle dV_{\omega_E} = 0\}.$$

- The following is an important ingredient in establishing the Hodge decomposition.

Theorem 4 (Poincaré inequality)

Let $p \leq n$ be a positive integer. There exists a positive constant $C > 0$ such that

$$\int_{X_0} |u|^2_{\omega_E} e^{-\varphi_L} dV \leq C \left(\int_{X_0} |\bar{\partial} u|^2_{\omega_E} e^{-\varphi_L} dV_{\omega_E} + \int_{X_0} \langle A \star u, \star u \rangle dV_{\omega_E} \right)$$

for any L-valued form u of type $(p, 0)$ which belongs to the domain of $\bar{\partial}$ and which is orthogonal to the space $H^{(p)}$.

- Application: same results hold for metrics with conic singularities along Y

$$\omega_{\mathcal{C}}|_\Omega = \sum_{i=1}^{r} \frac{\sqrt{-1} dz_i \wedge d\overline{z}_i}{|z_i|^{2\frac{m_i - 1}{m_i}}} + \sum_{i=r+1}^{n} \sqrt{-1} dz_i \wedge d\overline{z}_i.$$

End of the proof of Theorem 1

- We actually need the relative version of the Poincaré inequality.

Theorem 5

We assume that $D \ni t \mapsto \dim\left(\ker(\Delta''_t)\right)$ is constant. Then there exists $C > 0$:

$$\int_{X_t} |u|^2_{\omega_E} e^{-\varphi_L} dV_{\omega_E} \leq C \left(\int_{X_t} |\bar{\partial} u|^2_{\omega_E} e^{-\phi} dV_{\omega_E} + \int_{X_t} \langle A \star u, \star u \rangle dV_{\omega_E} \right)$$

for all L^2 forms u orthogonal to the space $H_t^{(p)}$ defined on the fiber X_t.

End of the proof of Theorem 1

- We actually need the relative version of the Poincaré inequality.

Theorem 5

We assume that $D \ni t \mapsto \dim\left(\ker(\Delta_t'')\right)$ is constant. Then there exists $C > 0$:

$$\int_{X_t} |u|^2_{\omega_E} e^{-\varphi_L} dV_{\omega_E} \leq C \left(\int_{X_t} |\bar{\partial} u|^2_{\omega_E} e^{-\phi} dV_{\omega_E} + \int_{X_t} \langle A \star u, \star u \rangle dV_{\omega_E} \right)$$

for all L^2 forms u orthogonal to the space $H_t^{(p)}$ defined on the fiber X_t.

- Theorem 1 is proved as follows:
  - Consider the representative $\mathbf{u}$ of u given by contraction with the canonical lifting of $\dfrac{\partial}{\partial t}$

End of the proof of Theorem 1

- We actually need the relative version of the Poincaré inequality.

Theorem 5

We assume that $D \ni t \mapsto \dim\big(\ker(\Delta_t'')\big)$ is constant. Then there exists $C > 0$:

$$\int_{X_t} |u|^2_{\omega_E} e^{-\varphi_L} dV_{\omega_E} \leq C \left(\int_{X_t} |\bar{\partial} u|^2_{\omega_E} e^{-\phi} dV_{\omega_E} + \int_{X_t} \langle A \star u, \star u \rangle dV_{\omega_E} \right)$$

for all L^2 forms u orthogonal to the space $H_t^{(p)}$ defined on the fiber X_t.

- Theorem 1 is proved as follows:
  - Consider the representative $\mathbf{u}$ of u given by contraction with the canonical lifting of $\dfrac{\partial}{\partial t}$
  - Since $D'_{\mathcal{F}} u = 0$, it follows that if $D'\mathbf{u} = dt \wedge \mu$ then $\mu|_{\mathcal{X}_t}$ is $\bar{\partial}^\star$-exact.

End of the proof of Theorem 1

- By Theorem 3 we can solve the fiberwise equation $\bar{\partial}^{\star}\beta_t = \mu|_{\mathcal{X}_t}$. Moreover, can assume that β_t orthogonal to $\mathrm{Ker}(\bar{\partial}^{\star})$.

End of the proof of Theorem 1

- By Theorem 3 we can solve the fiberwise equation $\bar{\partial}^{\star}\beta_t = \mu|_{\mathcal{X}_t}$. Moreover, can assume that β_t orthogonal to $\mathrm{Ker}(\bar{\partial}^{\star})$.
- The family $(\beta_t)_{t\in\mathbb{D}}$ is varying continuously with respect to t.

End of the proof of Theorem 1

- By Theorem 3 we can solve the fiberwise equation $\bar{\partial}^\star \beta_t = \mu|_{\mathcal{X}_t}$. Moreover, can assume that β_t orthogonal to $\mathrm{Ker}(\bar{\partial}^\star)$.
- The family $(\beta_t)_{t\in\mathbb{D}}$ is varying continuously with respect to t.
- We set $\mathbf{u}_1 := u - dt \wedge \star_t \beta_t$: it is the representative we are looking for.

End of the proof of Theorem 1

- By Theorem 3 we can solve the fiberwise equation $\bar{\partial}^\star \beta_t = \mu|_{\mathcal{X}_t}$. Moreover, can assume that β_t orthogonal to $\mathrm{Ker}(\bar{\partial}^\star)$.
- The family $(\beta_t)_{t\in\mathbb{D}}$ is varying continuously with respect to t.
- We set $\mathbf{u}_1 := u - dt \wedge \star_t \beta_t$: it is the representative we are looking for.

A remark

The continuity of $(\beta_t)_{t\in\mathbb{D}}$ is reasonably involved, based on Theorem 5 combined with standard arguments. Actually the statement should be that $(\beta_t)_{t\in\mathbb{D}}$ is a smooth family.

End of the proof of Theorem 1

- By Theorem 3 we can solve the fiberwise equation $\bar{\partial}^\star \beta_t = \mu|_{\mathcal{X}_t}$. Moreover, can assume that β_t orthogonal to $\mathrm{Ker}(\bar{\partial}^\star)$.
- The family $(\beta_t)_{t\in\mathbb{D}}$ is varying continuously with respect to t.
- We set $\mathbf{u}_1 := u - dt \wedge \star_t \beta_t$: it is the representative we are looking for.

A remark

The continuity of $(\beta_t)_{t\in\mathbb{D}}$ is reasonably involved, based on Theorem 5 combined with standard arguments. Actually the statement should be that $(\beta_t)_{t\in\mathbb{D}}$ is a smooth family.

- In what follows we discuss the proof of Theorem 2.

A version of Theorem 2

- We consider the following setting:

A version of Theorem 2

- We consider the following setting:
  - The curvature current of (L, h_L) can be written as

$$\sqrt{-1}\Theta(L, h_L) = \omega_L + [F]$$

A version of Theorem 2

- We consider the following setting:
  - The curvature current of (L, h_L) can be written as

$$\sqrt{-1}\Theta(L, h_L) = \omega_L + [F]$$

  - F effective, its support is snc and transverse to fibers

A version of Theorem 2

- We consider the following setting:
  - The curvature current of (L, h_L) can be written as
  $$\sqrt{-1}\Theta(L, h_L) = \omega_L + [F]$$
  - F effective, its support is snc and transverse to fibers
  - The smooth form $\omega_L \geq 0$ on $\mathcal{X}$ and fiber-wise Kähler.

A version of Theorem 2

- We consider the following setting:
  - The curvature current of (L, h_L) can be written as

$$\sqrt{-1}\Theta(L, h_L) = \omega_L + [F]$$

  - F effective, its support is snc and transverse to fibers
  - The smooth form $\omega_L \geq 0$ on $\mathcal{X}$ and fiber-wise Kähler.
- We have the following result.

Theorem 2'

Consider $p : \mathcal{X} \to \mathbb{D}$ and $(L, h_L) \to \mathcal{X}$ as above. For every $u \in H^0(\mathbb{D}, \mathcal{F})$ and any $t \in \mathbb{D}$

A version of Theorem 2

- We consider the following setting:
  - The curvature current of (L, h_L) can be written as
  $$\sqrt{-1}\Theta(L, h_L) = \omega_L + [F]$$
  - F effective, its support is snc and transverse to fibers
  - The smooth form $\omega_L \geq 0$ on $\mathcal{X}$ and fiber-wise Kähler.
- We have the following result.

Theorem 2'

Consider $p : \mathcal{X} \to \mathbb{D}$ and $(L, h_L) \to \mathcal{X}$ as above. For every $u \in H^0(\mathbb{D}, \mathcal{F})$ and any $t \in \mathbb{D}$

$$\langle \sqrt{-1}\Theta_{h_{\mathcal{F}}}(\mathcal{F})u, u\rangle_t \geq c_n \int_{X_t} c(\omega_L) u \wedge \bar{u} e^{-\varphi_L}.$$

Approximation

- We proceed by approximation

Approximation

- We proceed by approximation

$$\omega_{L,\varepsilon} := \omega_L + \delta_\varepsilon \sqrt{-1} dt \wedge d\bar{t} - \sqrt{-1}\varepsilon \sum_i \partial\bar{\partial} \log\log \frac{1}{|s_i|^2}$$

Approximation

- We proceed by approximation

$$\omega_{L,\varepsilon} := \omega_L + \delta_\varepsilon \sqrt{-1} dt \wedge d\bar{t} - \sqrt{-1}\varepsilon \sum_i \partial\bar{\partial} \log\log \frac{1}{|s_i|^2}$$

- A few properties:

Approximation

- We proceed by approximation

$$\omega_{L,\varepsilon} := \omega_L + \delta_\varepsilon \sqrt{-1} dt \wedge d\bar{t} - \sqrt{-1}\varepsilon \sum_i \partial\bar{\partial} \log\log \frac{1}{|s_i|^2}$$

- A few properties:
  - There exists $C > 0$ such that $c(\omega_{L,\varepsilon}) \leq C$ and $\lim_{\varepsilon \to 0} c(\omega_{L,\varepsilon}) = c(\omega_L)$.

Approximation

- We proceed by approximation

$$\omega_{L,\varepsilon} := \omega_L + \delta_\varepsilon \sqrt{-1} dt \wedge d\bar{t} - \sqrt{-1}\varepsilon \sum_i \partial\bar{\partial} \log\log \frac{1}{|s_i|^2}$$

- A few properties:
  - There exists $C > 0$ such that $c(\omega_{L,\varepsilon}) \leq C$ and $\lim_{\varepsilon \to 0} c(\omega_{L,\varepsilon}) = c(\omega_L)$.
  - Let $h_{L,\varepsilon}$ be the metric on L whose curvature in the complement of F is $\omega_{L,\varepsilon}$. Then $\mathcal{I}(h_{L,\varepsilon}) = \mathcal{I}(h_L)$.

Approximation

- We proceed by approximation

$$\omega_{L,\varepsilon} := \omega_L + \delta_\varepsilon \sqrt{-1} dt \wedge d\bar{t} - \sqrt{-1}\varepsilon \sum_i \partial\bar{\partial} \log\log \frac{1}{|s_i|^2}$$

- A few properties:
  - There exists $C > 0$ such that $c(\omega_{L,\varepsilon}) \leq C$ and $\lim_{\varepsilon \to 0} c(\omega_{L,\varepsilon}) = c(\omega_L)$.
  - Let $h_{L,\varepsilon}$ be the metric on L whose curvature in the complement of F is $\omega_{L,\varepsilon}$. Then $\mathcal{I}(h_{L,\varepsilon}) = \mathcal{I}(h_L)$.

- Let $\mathbf{u}_\varepsilon$ be the representative of u given by $V_\varepsilon \rfloor (dt \wedge U_0)$.

Approximation

- We proceed by approximation

$$\omega_{L,\varepsilon} := \omega_L + \delta_\varepsilon \sqrt{-1} dt \wedge d\bar{t} - \sqrt{-1}\varepsilon \sum_i \partial\bar{\partial} \log\log \frac{1}{|s_i|^2}$$

- A few properties:
  - There exists $C > 0$ such that $c(\omega_{L,\varepsilon}) \leq C$ and $\lim_{\varepsilon \to 0} c(\omega_{L,\varepsilon}) = c(\omega_L)$.
  - Let $h_{L,\varepsilon}$ be the metric on L whose curvature in the complement of F is $\omega_{L,\varepsilon}$. Then $\mathcal{I}(h_{L,\varepsilon}) = \mathcal{I}(h_L)$.

- Let $\mathbf{u}_\varepsilon$ be the representative of u given by $V_\varepsilon \rfloor (dt \wedge U_0)$.
- We introduce: $\bar{\partial}\mathbf{u}_\varepsilon = dt \wedge \eta_\varepsilon$ and $D'\mathbf{u}_\varepsilon = dt \wedge \mu_\varepsilon$.

End of the proof

- Our current context: $(X, \omega_{L,\varepsilon})$ and $(L, h_{L,\varepsilon})$

End of the proof

- Our current context: $(X, \omega_{L,\varepsilon})$ and $(L, h_{L,\varepsilon})$
  - The curvature formula looks better:

$$-\frac{\partial^2}{\partial t \partial \bar{t}}(\|u\|^2_{h_{\mathcal{F},\varepsilon}}) = c_n \int_{X_t} c(\omega_\varepsilon)\mathbf{u}_\varepsilon \wedge \bar{\mathbf{u}}_\varepsilon e^{-\varphi_\varepsilon} + \int_{X_t} |\eta_\varepsilon|^2 e^{-\varphi_\varepsilon} dV_{\omega_\varepsilon} - \int_{X_t} |\mu_\varepsilon|^2 e^{-\varphi_\varepsilon} dV_{\omega_\varepsilon}$$

End of the proof

- Our current context: $(X, \omega_{L,\varepsilon})$ and $(L, h_{L,\varepsilon})$
  - The curvature formula looks better:

$$-\frac{\partial^2}{\partial t \partial \bar{t}}(\|u\|^2_{h_{\mathcal{F},\varepsilon}}) = c_n \int_{X_t} c(\omega_\varepsilon)\mathbf{u}_\varepsilon \wedge \bar{\mathbf{u}}_\varepsilon e^{-\varphi_\varepsilon} + \int_{X_t} |\eta_\varepsilon|^2 e^{-\varphi_\varepsilon} dV_{\omega_\varepsilon} - \int_{X_t} |\mu_\varepsilon|^2 e^{-\varphi_\varepsilon} dV_{\omega_\varepsilon}$$

  - The LHS is equal to

$$\langle \sqrt{-1}\Theta_{h_{\mathcal{F},\varepsilon}}(\mathcal{F})u, u\rangle - \|P(\mu_\varepsilon)\|^2$$

End of the proof

- Our current context: $(X, \omega_{L,\varepsilon})$ and $(L, h_{L,\varepsilon})$
  - The curvature formula looks better:

$$-\frac{\partial^2}{\partial t \partial \bar{t}}(\|u\|^2_{h_{\mathcal{F},\varepsilon}}) = c_n \int_{X_t} c(\omega_\varepsilon)\mathbf{u}_\varepsilon \wedge \bar{\mathbf{u}}_\varepsilon e^{-\varphi_\varepsilon} + \int_{X_t} |\eta_\varepsilon|^2 e^{-\varphi_\varepsilon} dV_{\omega_\varepsilon} - \int_{X_t} |\mu_\varepsilon|^2 e^{-\varphi_\varepsilon} dV_{\omega_\varepsilon}$$

  - The LHS is equal to

$$\langle \sqrt{-1}\Theta_{h_{\mathcal{F},\varepsilon}}(\mathcal{F})u, u \rangle - \|P(\mu_\varepsilon)\|^2$$

  - We have $\bar{\partial}\mu_\varepsilon = D'\eta_\varepsilon$ which combined with L^2 estimates:

$$\int_{X_t} |\mu_\varepsilon^\perp|^2_{\omega_\varepsilon} e^{-\varphi_\varepsilon} dV_{\omega_\varepsilon} \leq \int_{X_t} |\eta_\varepsilon|^2_{\omega_\varepsilon} e^{-\varphi_\varepsilon} dV_{\omega_\varepsilon}$$

End of the proof

- Our current context: $(X, \omega_{L,\varepsilon})$ and $(L, h_{L,\varepsilon})$
  - The curvature formula looks better:

$$-\frac{\partial^2}{\partial t \partial \bar{t}}(\|u\|^2_{h_{\mathcal{F},\varepsilon}}) = c_n \int_{X_t} c(\omega_\varepsilon)\mathbf{u}_\varepsilon \wedge \bar{\mathbf{u}}_\varepsilon e^{-\varphi_\varepsilon} + \int_{X_t} |\eta_\varepsilon|^2 e^{-\varphi_\varepsilon} dV_{\omega_\varepsilon} - \int_{X_t} |\mu_\varepsilon|^2 e^{-\varphi_\varepsilon} dV_{\omega_\varepsilon}$$

  - The LHS is equal to

$$\langle \sqrt{-1}\Theta_{h_{\mathcal{F},\varepsilon}}(\mathcal{F})u, u\rangle - \|P(\mu_\varepsilon)\|^2$$

  - We have $\bar{\partial}\mu_\varepsilon = D'\eta_\varepsilon$ which combined with L^2 estimates:

$$\int_{X_t} |\mu_\varepsilon^\perp|^2_{\omega_\varepsilon} e^{-\varphi_\varepsilon} dV_{\omega_\varepsilon} \leq \int_{X_t} |\eta_\varepsilon|^2_{\omega_\varepsilon} e^{-\varphi_\varepsilon} dV_{\omega_\varepsilon}$$

  - The decomposition $\mu_\varepsilon = P(\mu_\varepsilon) + \mu_\varepsilon^\perp$ is orthogonal and we are done.