Global pluripotential theory over a trivially valued field

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Goal: develop potential theory using the trivial absolute value on $\mathbb{C}$.

$$|a| = 1 \qquad a \in \mathbb{C}^*$$
$$\| $$
$$\lim_{\rho \to 0} |a|_\infty^\rho$$

Motivation: K-stability, degeneration problems.

Rmk: some new aspects also in holo case

The complex story

X = (smooth) proj. complex var.
L = ample lb

$$\begin{array}{l} PSH := \{\text{semipos. sing. metrics on } L\} \\ \quad \wr\wr \\ PSH(X,\omega) \qquad \omega = \Theta(h) \in c_1(L) \end{array}$$

<u>Ex</u>: $\varphi = \frac{1}{2m} \log \sum_1^N |s_j|_h^2 \qquad s_j \in H^0(X, mL)$ no common zero.

<u>Thm</u> [Demailly] $PSH = \{\text{decreasing limits of such } \varphi\}$.

<u>Energy classes</u> [Cegrell, Guedj–Zeriahi, BBEGZ and subsets, ...]

$$\underbrace{\mathcal{H}}_{\text{as above}} \subset \underbrace{CPSH}_{\text{cont}} \subset \mathcal{E}^1 \subset \ldots \subset PSH$$

$\mathcal{E}^1 := \{E > -\infty\}$.

$$E(\varphi) = \frac{1}{n+1} \sum_{j=0}^n \frac{1}{(L^n)} \int_X \varphi\, \omega_\varphi^j \wedge \omega^{n-j} \qquad \varphi \in \mathcal{H}$$

$$\mathcal{M}^1 := \{E^\vee < \infty\} \subset \{\text{prob. measures on } X\}$$

$$E^\vee(\mu) = \sup\left\{ E(\varphi) - \int \varphi \mu \;\middle|\; \varphi \in \mathcal{H} \right\}.$$

$$MA: \quad \begin{array}{ccc} \mathcal{E}^1/\mathbb{R} & \xrightarrow{\sim} & \mathcal{M}^1 \\ \varphi & \longmapsto & \omega_\varphi^n \end{array} \qquad \text{(homeo in strong top.)}$$

<u>Envelope</u>: $P: C^0(X) \longrightarrow CPSH(X)$

$$P(f) := \sup\{\varphi \in PSH \mid \varphi \le f\}.$$

Darvas metric :

$$d_1(\varphi,\psi) = E(\varphi)+E(\psi) - 2E(P(\varphi\wedge\psi))$$

complete metric on $\mathcal{E}^1$, defines strong topology.

Singular case : everything above works if X is normal or merely unibranch

ok   not ok.

Rmk : can define "everything" above using sections $s \in H^0(X, mL)$

Toric case : $M, N \simeq \mathbb{Z}^n$ dual lattices

$P \subset M_{\mathbb{R}} \simeq \mathbb{R}^n$ polytope

$PSH^{tor} \simeq \{$convex fcns on $g : P \to \mathbb{R}\}$

$\underset{Leg}{\simeq} \{$ convex fcns $f : N_{\mathbb{R}} \to \mathbb{R}$ w. growth cond'n $\}$.

$$\mathcal{E}^{1,tor} = L^1(P) \cap Conv(P).$$

$MA \simeq$ real Monge-Ampère.

Trivially valued case

$\mathbb{C}$, $|a|=1$ $a\in\mathbb{C}^*$

X cplx proj. var.

L ample lb.

X^{an} = Berkovich analytification compact Hausdorff

$$X^{an} \supset X^{val} \supset X^{div} \qquad Y \supset E$$

valuations $v:\mathbb{C}(X)\to\mathbb{R}$ divisorial vals

$$Y \to X$$

$$v = c\cdot \mathrm{ord}_E$$

$$s\in H^0(X,mL) \rightsquigarrow |s|\in C^0(X^{an})$$
$$|s|(v) = e^{-v(s)}$$

$$\mathcal{H} := \left\{ \max_j \frac{1}{m}(\log|s_j| + \lambda_j) \;\middle|\; \underset{\text{no common zero}}{s_j\in H^0(X,mL)} \quad \lambda_j\in\mathbb{Z} \right\}.$$

$PSH :=$ {decr. limits in $\mathcal{H}$}.

(topology: ptwise conv. on X^{div})

Toric case:

$PSH^{tor} \simeq$ {convex fcns on $P\subset M_{\mathbb{R}}$}.

$\simeq$ {conv. fcns on $N_{\mathbb{R}}\simeq\mathbb{R}^n$ w. growth cond's}.

($N_{\mathbb{R}}\subset X^{val}\subset X^{an}$ monomial vals)

Examples

① $u \in S$ (singularity type of) semipos. metric on L (usual sense)

$\rightsquigarrow \varphi \in PSH$ (triv. valued sense)

$$\varphi(c\cdot \mathrm{ord}_E) = -c\cdot \underbrace{\mathrm{ord}_E(u\circ\pi)}_{\text{gen. Lelong no.}}$$

$$\begin{array}{c} Y \supset E \\ \downarrow \pi \\ X \end{array}$$

② $(\mathcal{X},\mathcal{L})$ test configurations.

$\rightsquigarrow \varphi_{(\mathcal{X},\mathcal{L})} \in \mathcal{H} \subset PSH$

③ $(u_t)_0^\infty$ geodesic ray in PSH^{hol} $X\times\mathbb{D}$

$\rightsquigarrow \varphi \in PSH^{triv.val.}$ Lelong nos along $X\times\{0\}$.

Continuity of envelopes

$$P(f) := \sup\{\varphi \in PSH \mid \varphi \le f\}.$$

TFAE:

(1) $f \in C^0(X^{an}) \Rightarrow P(f) \in CPSH$ CoE

(2) every increasing upper bdd seq in PSH converges. $u_1 \le u_2 \le \dots$

Thm: CoE holds when X is smooth.

Conj: —— " —— is unibranch.

Thm: CoE $\Rightarrow$ if $E \subset X^{an}$ then E pluripolar $\Longleftrightarrow$ E negligeable

Monge-Ampère energy / operator

Can define: $\quad E : PSH \longrightarrow \mathbb{R} \cup \{-\infty\}$

$$\mathcal{E}^1 := \{E > -\infty\}$$

$$\mathcal{M}^1 := \{E^\vee < \infty\}$$

$$MA : \mathcal{E}^1/\mathbb{R} \longrightarrow \mathcal{M}^1$$

First define E, MA on $\mathcal{H}$, then approximate.

Approach 1: Intersection theory on tc's.

Approach 2: Chambert-Loir -- Ducros, Lagerberg forms, currents on Berk. spaces.

Ex: toric case. $\mathcal{E}^1 \simeq \mathrm{Conv}(P) \cap L^1(P)$

MA = real MA on $N_{\mathbb{R}} \simeq \mathbb{R}^n \subset X^{an}$.

Thm: (1) $MA : \mathcal{E}^1/\mathbb{R} \longrightarrow \mathcal{M}^1$ strongly cont, dense image

(2) surjective iff CoE holds.

$$d_1(\varphi,\psi) := \inf\{E(\varphi)+E(\psi)-2E(\chi) \mid \chi \le \varphi,\psi\}.$$

Thm: (1) d_1 well-def metric on $\mathcal{E}^1$ def. strong top.

(2) complete iff CoE holds

(3) geodesics [Reboulet]

d_1 induces (pseudo)metric $\underline{d_1}$ on $\mathcal{E}^1/\mathbb{R} \hookrightarrow \mathcal{M}^1$

Thm: $\underline{d_1}$ extends uniquely to a complete metric on $\mathcal{M}^1$

Application to K-stability

X = smooth (klt)

$M : \mathcal{M}^1 \longrightarrow \mathbb{R} \cup \{\infty\}$ Mabuchi) NA

= entropy + energy.

Say M coercive if $M \geq \varepsilon \cdot E^\vee$ on $\mathcal{M}^1$

Thm $\underbrace{\overbrace{[\text{C.Li}]}^{\text{Cheng-Cheng}}\ M_{NA} \text{ coercive}} \Rightarrow \underbrace{\exists \text{ cscK metric}}_{\text{hol}}$.

Thm [BJ, in progress].
Coercivity is an open condition on $L \in \mathrm{Amp}(X)$