Holomorphic sections of line bundles vanishing along subvarieties

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Plan of the talk:

1. Preliminaries and notation
2. Dimension of spaces of holomorphic sections vanishing along subvarieties
3. Envelopes of quasiplurisubharmonic functions with poles along a divisor
4. Convergence of Fubini-Study currents
5. Zeros of random sequences of holomorphic sections

1. Preliminaries and notation

X compact complex manifold, $\dim X = n$, ω Hermitian form on X

$$d = \partial + \overline{\partial}\,,\ \ d^c = \frac{1}{2\pi i}(\partial - \overline{\partial})\,,\ \ dd^c = \frac{i}{\pi}\,\partial\overline{\partial}$$

If α is a smooth real closed $(1,1)$-form on X, we let

$$\mathrm{PSH}(X,\alpha) = \{\varphi :\ X \to \mathbb{R} \cup \{-\infty\} :\ \varphi \text{ qpsh},\ \alpha + dd^c\varphi \geq 0\}.$$

$\nu(\varphi, x) = \nu(\alpha + dd^c\varphi, x)$ is the Lelong number of φ at $x \in X$

$H^{1,1}_{\partial\overline{\partial}}(X,\mathbb{R}) := \{\{\alpha\}_{\partial\overline{\partial}} :\ \alpha$ smooth real closed $(1,1)$-form on $X\}$

Since X is compact, $H^{1,1}_{\partial\overline{\partial}}(X,\mathbb{R})$ is finite dimensional. A class $\{\alpha\}_{\partial\overline{\partial}}$ is *big* if it contains a *Kähler current* (positive closed $(1,1)$ current $T \geq \varepsilon\omega$).

If $\{\alpha\}_{\partial\overline{\partial}}$ is big then, by Demailly's regularization theorem,

$\exists\, T \in \{\alpha\}_{\partial\overline{\partial}}$ Kähler current with *analytic singularities*, i.e.

$T = \alpha + dd^c\varphi \geq \epsilon\omega$, where $\epsilon > 0$, $\varphi = c\log\left(\sum_{j=1}^{N} |g_j|^2\right) + \chi$ locally on X, with $c > 0$, χ a smooth function and g_j holomorphic functions.

Non-ample locus of $\{\alpha\}_{\partial\overline{\partial}}$:

$$\mathrm{NAmp}\big(\{\alpha\}_{\partial\overline{\partial}}\big) = \bigcap \big\{E_+(T) :\ T \in \{\alpha\}_{\partial\overline{\partial}} \text{ Kähler current}\big\},$$

where $E_+(T) = \{x \in X : \nu(T,x) > 0\}$.

Bouckssom: $\exists\, T \in \{\alpha\}_{\partial\overline{\partial}}$ Kähler current with analytic singularities such that $E_+(T) = \mathrm{NAmp}\big(\{\alpha\}_{\partial\overline{\partial}}\big)$.

Hence $\mathrm{NAmp}\big(\{\alpha\}_{\partial\overline{\partial}}\big)$ is an analytic subset of X.

Plurisubharmonic (psh) functions and currents on analytic spaces

X irreducible complex space, $\dim X = n$, X_{reg}, X_{sing}

$\varphi : X \to [-\infty, \infty)$ is *(strictly) psh* if for every $x \in X$ there exist:

$\tau : U_x \subset X \to G \subset \mathbb{C}^N$ local embedding, and

$\widetilde{\varphi} : G \to [-\infty, \infty)$ (strictly) psh with $\varphi|_{U_x} = \widetilde{\varphi} \circ \tau$.

Set $\Omega^\infty_{p,q}(U) := \tau^*(\Omega^\infty_{p,q}(G))$, where $\tau^* : \Omega^\infty_{p,q}(G) \to \Omega^\infty_{p,q}(U \cap X_{\mathrm{reg}})$.

$\mathcal{D}_{p,q}(X) \subset \Omega^\infty_{p,q}(X)$: space of smooth (p, q)-forms with compact support.

Dual $\mathcal{D}'_{p,q}(X)$: space of currents of bidegree $(n-p, n-q)$ on X.

$\mathcal{T}(X) \subset \mathcal{D}'_{n-1,n-1}(X)$ positive closed currents with local psh potentials.

Kähler form on X: $\omega \in \mathcal{T}(X)$ with smooth strictly psh local potentials.

Singular Hermitian holomorphic line bundles

X compact, irreducible, normal complex space, $\dim X = n$

$\pi : L \longrightarrow X$ holomorphic line bundle on X:

$X = \bigcup U_\alpha$, U_α open, $g_{\alpha\beta} \in \mathcal{O}^*_X(U_\alpha \cap U_\beta)$ are the *transition functions*.

$H^0(X, L) =$ space of global holomorphic sections of L, $\dim H^0(X, L) < \infty$

Singular Hermitian metric h on L: $\{\varphi_\alpha \in L^1_{loc}(U_\alpha, \omega^n)\}_\alpha$ such that

$\varphi_\alpha = \varphi_\beta + \log |g_{\alpha\beta}|$ on $U_\alpha \cap U_\beta$, $|e_\alpha|_h = e^{-\varphi_\alpha}$ (e_α local frame on U_α).

The curvature current $c_1(L, h) \in \mathcal{D}'_{n-1,n-1}(X)$ of h:

$$c_1(L, h) = dd^c \varphi_\alpha \text{ on } U_\alpha \cap X_{\rm reg}.$$

If $c_1(L, h) \geq 0$ then φ_α is psh on $U_\alpha \cap X_{\rm reg}$, hence on U_α (X is normal). So

$$c_1(L, h) \in \mathcal{T}(X).$$

2. Dimension of spaces of holomorphic sections vanishing along subvarieties

X compact complex manifold, $\dim X = n$, $L \to X$ holomorphic line bundle

$L^p := L^{\otimes p}$, $\quad H^0(X, L^p) =$ space of global holomorphic sections of L^p

Siegel's Lemma: $\exists\, C > 0$ *such that* $\dim H^0(X, L^p) \leq Cp^n$ *for all* $p \geq 1$.

A line bundle L is called *big* if $\limsup_{p\to\infty} \frac{1}{p^n} \dim H^0(X, L^p) > 0$.

If L is big one can show that $\exists\, c > 0$, $p_0 \geq 1$, such that

$$\dim H^0(X, L^p) \geq cp^n \text{ for all } p \geq p_0.$$

Ji-Shiffman: *L is big if and only if $\exists\, h$ singular Hermitian metric on L such that $c_1(L, h)$ is a Kähler current.*

(A) X is a compact, irreducible, normal complex space, $\dim X = n$.

(B) L is a holomorphic line bundle on X.

(C) $\Sigma = (\Sigma_1, \ldots, \Sigma_\ell)$, $\Sigma_j \not\subset X_{\rm sing}$, are distinct irreducible proper analytic subsets of X.

(D) $\tau = (\tau_1, \ldots, \tau_\ell)$, $\tau_j \in (0, +\infty)$, and $\tau_j > \tau_k$ if $\Sigma_j \subset \Sigma_k$.

$H^0_0(X, L^p)$ space of sections vanishing to order $\geq \tau_j p$ along Σ_j, $1 \leq j \leq \ell$

$$t_{j,p} = \begin{cases} \tau_j p & \text{if } \tau_j p \in \mathbb{N} \\ \lfloor \tau_j p \rfloor + 1 & \text{if } \tau_j p \notin \mathbb{N} \end{cases}, \ \ 1 \leq j \leq \ell, \ \ p \geq 1\,.$$

$$H^0_0(X, L^p) = H^0_0(X, L^p, \Sigma, \tau) := \left\{ S \in H^0(X, L^p) :\, \operatorname{ord}(S, \Sigma_j) \geq t_{j,p} \right\}$$

We say that **the triplet** (L, Σ, τ) **is big** if $\displaystyle\limsup_{p\to\infty} \frac{1}{p^n} \dim H^0_0(X, L^p) > 0$.

Characterization when X is a complex manifold and $\dim \Sigma_j = n - 1$:

Theorem 1

Let: X compact complex manifold, $\dim X = n$, *L holomorphic line bundle on X, $\Sigma = (\Sigma_1, \ldots, \Sigma_\ell)$, $\tau = (\tau_1, \ldots, \tau_\ell)$, where Σ_j are distinct irreducible complex hypersurfaces in X, $\tau_j \in (0, +\infty)$. The following are equivalent:*

(i) (L, Σ, τ) is big;

(ii) There exists a singular Hermitian metric h on L such that

$$c_1(L,h) - \sum_{j=1}^{\ell} \tau_j [\Sigma_j]$$

is a Kähler current on X, where $[\Sigma_j]$ is the current of integration along Σ_j;

(iii) $\exists\, c > 0$, $p_0 \geq 1$, such that $\dim H^0_0(X, L^p) \geq cp^n$ *for all $p \geq p_0$.*

Proposition 2

Let X*,* Σ *verify (A), (C). Then there exist a compact complex manifold* $\widetilde{X}$*,* $\dim\widetilde{X}=n$*, and a surjective holomorphic map* $\pi:\widetilde{X}\to X$*, given as the composition of finitely many blow-ups with smooth center, such that:*

(i) $\exists\, Y\subset X$ *analytic subset such that* $\dim Y\leq n-2$, $X_{\rm sing}\subset Y$, $\Sigma_j\subset Y$ *if* $\dim\Sigma_j\leq n-2$, $Y\subset X_{\rm sing}\cup\bigcup_{j=1}^{\ell}\Sigma_j$, $E=\pi^{-1}(Y)$ *is a divisor in* $\widetilde{X}$ *that has only normal crossings, and* $\pi:\widetilde{X}\setminus E\to X\setminus Y$ *is a biholomorphism.*

(ii) There exist smooth complex hypersurfaces $\widetilde{\Sigma}_1,\ldots,\widetilde{\Sigma}_\ell$ *in* $\widetilde{X}$ *such that* $\pi(\widetilde{\Sigma}_j)=\Sigma_j$*. If* $\dim\Sigma_j=n-1$ *then* $\widetilde{\Sigma}_j$ *is the final strict transform of* Σ_j*, and if* $\dim\Sigma_j\leq n-2$ *then* $\widetilde{\Sigma}_j$ *is an irreducible component of* E*.*

(iii) If $F\to X$ *is a holomorphic line bundle and* $S\in H^0(X,F)$ *then* $\operatorname{ord}(S,\Sigma_j)=\operatorname{ord}(\pi^\star S,\widetilde{\Sigma}_j)$*, for all* $j=1,\ldots,\ell$*.*

If $\widetilde{X}$, π, $\widetilde{\Sigma} := (\widetilde{\Sigma}_1, \ldots, \widetilde{\Sigma}_\ell)$, verify the conclusions of Proposition 2, we say that $(\widetilde{X}, \pi, \widetilde{\Sigma})$ is a **divisorization** of (X, Σ).

Theorem 3

Let X, L, Σ, τ verify assumptions (A)-(D). The following are equivalent:

(i) (L, Σ, τ) is big;

(ii) $\forall\, (\widetilde{X}, \pi, \widetilde{\Sigma})$ divisorization of (X, Σ), $\exists\, h^\star$ singular metric on $\pi^\star L$ such that $c_1(\pi^\star L, h^\star) - \sum_{j=1}^{\ell} \tau_j [\widetilde{\Sigma}_j]$ is a Kähler current on $\widetilde{X}$;

(iii) Assertion (ii) holds for some divisorization $(\widetilde{X}, \pi, \widetilde{\Sigma})$ of (X, Σ);

(iv) $\exists\, c > 0$, $p_0 \geq 1$, such that $\dim H^0_0(X, L^p) \geq cp^n$ for all $p \geq p_0$.

Theorem 3 follows directly from Theorem 1 since, by Proposition 2,

$$H^0_0(X, L^p, \Sigma, \tau) \cong H^0_0(\widetilde{X}, \pi^\star L^p, \widetilde{\Sigma}, \tau)\,,\ \ \forall\, p \geq 1.$$

Theorem 3 has the following interesting corollary:

Corollary 4

Let X, L, Σ, τ verify assumptions (A)-(D). Assume that $\dim \Sigma_j = n - 1$ and let $\Sigma'_j \subset \Sigma_j$ be distinct irreducible proper analytic subsets such that $\Sigma'_j \not\subset X_{\rm sing}$, $j = 1, \ldots, \ell$. For $\delta > 0$ set

$$\Sigma' = (\Sigma_1, \ldots, \Sigma_\ell, \Sigma'_1, \ldots, \Sigma'_\ell),\ \ \tau' = (\tau_1, \ldots, \tau_\ell, \tau_1 + \delta, \ldots, \tau_\ell + \delta).$$

If (L, Σ, τ) is big, then (L, Σ', τ') is big, for all $\delta > 0$ sufficiently small.

3. Envelopes of qpsh functions with poles along a divisor

X compact complex manifold, $\dim X = n$, ω Hermitian form on X,
dist $=$ distance on X induced by ω

$\Sigma_j \subset X$ irreducible complex hypersurfaces, $\tau_j > 0$, where $1 \leq j \leq \ell$.
Write $\Sigma = (\Sigma_1, \ldots, \Sigma_\ell)$, $\tau = (\tau_1, \ldots, \tau_\ell)$,

Let: α be a smooth closed real $(1,1)$-form on X,
g_j smooth Hermitian metric on $\mathscr{O}_X(\Sigma_j)$,
s_{Σ_j} be the canonical section of $\mathscr{O}_X(\Sigma_j)$, $1 \leq j \leq \ell$,

$$\beta_j = c_1(\mathscr{O}_X(\Sigma_j), g_j)\,,\ \ \theta = \alpha - \sum_{j=1}^{\ell} \tau_j \beta_j\,,\ \ \sigma_j := |s_{\Sigma_j}|_{g_j}\,.$$

Lelong-Poincaré Formula: $[\Sigma_j] = \beta_j + dd^c \log \sigma_j$

$$\mathcal{L}(X, \alpha, \Sigma, \tau) = \{\psi \in \mathrm{PSH}(X, \alpha) :\ \nu(\psi, x) \geq \tau_j,\ \forall\, x \in \Sigma_j,\ 1 \leq j \leq \ell\}$$

Given $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ we consider the following:

$$\mathcal{A}(X, \alpha, \Sigma, \tau, \varphi) = \{\psi \in \mathcal{L}(X, \alpha, \Sigma, \tau) : \psi \leq \varphi \text{ on } X\}$$

$$\mathcal{A}'(X, \alpha, \Sigma, \tau, \varphi) =$$
$$= \Big\{\psi' \in \mathrm{PSH}(X, \theta) : \psi' \leq \varphi - \sum_{j=1}^{\ell} \tau_j \log \sigma_j \text{ on } X \setminus \bigcup_{j=1}^{\ell} \Sigma_j \Big\}$$

$$\begin{aligned} \varphi_{\mathrm{eq}} = \varphi_{\mathrm{eq},\Sigma,\tau} &= \sup\{\psi : \psi \in \mathcal{A}(X, \alpha, \Sigma, \tau, \varphi)\} \\ &= \textit{ equilibrium envelope of } (\alpha, \Sigma, \tau, \varphi) \\ \varphi_{\mathrm{req}} = \varphi_{\mathrm{req},\Sigma,\tau} &= \sup\{\psi' : \psi' \in \mathcal{A}'(X, \alpha, \Sigma, \tau, \varphi)\} \\ &= \textit{ reduced equilibrium envelope of } (\alpha, \Sigma, \tau, \varphi) \end{aligned}$$

Motivated by the notion of *equilibrium metric* associated to a Hermitian metric on a holomorphic line bundle (Berman, Ross-Witt Nyström).

Proposition 5

Let $X, \Sigma, \tau, \alpha, \theta$ *be as above, and let* $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ *be an upper semicontinuous function. Then the following hold:*

(i) $\mathrm{PSH}(X, \theta) \ni \psi' \mapsto \psi := \psi' + \sum_{j=1}^{\ell} \tau_j \log \sigma_j \in \mathcal{L}(X, \alpha, \Sigma, \tau)$ *is well defined and bijective.*

(ii) $\exists\, C > 0$ *such that* $\sup_X \psi' \leq \sup_X \varphi + C$, $\forall\, \psi' \in \mathcal{A}'(X, \alpha, \Sigma, \tau, \varphi)$.

(iii) $\mathcal{A}(X, \alpha, \Sigma, \tau, \varphi) \neq \emptyset$ *if and only if* $\mathcal{A}'(X, \alpha, \Sigma, \tau, \varphi) \neq \emptyset$. *In this case,*

$$\varphi_{\mathrm{req}} \in \mathcal{A}'(X, \alpha, \Sigma, \tau, \varphi),\ \varphi_{\mathrm{eq}} \in \mathcal{A}(X, \alpha, \Sigma, \tau, \varphi),\ \varphi_{\mathrm{eq}} = \varphi_{\mathrm{req}} + \sum_{j=1}^{\ell} \tau_j \log \sigma_j .$$

(iv) If φ *is bounded and there exists a bounded* θ*-psh function, then* φ_{req} *is bounded on* X.

(v) If $\mathrm{PSH}(X, \theta) \neq \emptyset$ *and* $\varphi_1, \varphi_2 : X \to \mathbb{R}$ *are u.s.c. and bounded, then* $\varphi_{1,\mathrm{req}} - \sup_X |\varphi_1 - \varphi_2| \leq \varphi_{2,\mathrm{req}} \leq \varphi_{1,\mathrm{req}} + \sup_X |\varphi_1 - \varphi_2|$ *holds on* X.

Regularity properties of the equilibrium envelopes

Definition 6

$\phi: X \to [-\infty, \infty)$ is *Hölder with singularities along a proper analytic subset* $A \subset X$ if there exist constants $c, \varrho > 0$ and $0 < \nu \leq 1$ such that

$$|\phi(z) - \phi(w)| \leq \frac{c \operatorname{dist}(z, w)^{\nu}}{\min\{\operatorname{dist}(z, A), \operatorname{dist}(w, A)\}^{\varrho}}\,, \ \forall\, z, w \in X \setminus A\,.$$

We use the regularization techniques developed by Demailly, Berman-Demailly.

They were employed to study the regularity of envelopes of Hölder continuous or Lipschitz functions by Dinh-Ma-Nguyên, Ross-Witt Nyström, Darvas-Rubinstein.

Theorem 7

Let (X, ω) be a compact Hermitian manifold of dimension n, $\Sigma_j \subset X$ be irreducible complex hypersurfaces, and let $\tau_j > 0$, where $1 \leq j \leq \ell$. Let α be a smooth closed real $(1,1)$-form on X and

$$\theta = \alpha - \sum_{j=1}^{\ell} \tau_j \beta_j \,, \ \text{ where } \beta_j = c_1(\mathscr{O}_X(\Sigma_j), g_j) \,.$$

Assume that the class $\{\theta\}_{\partial\overline{\partial}}$ is big and let $Z_0 := \mathrm{NAmp}\big(\{\theta\}_{\partial\overline{\partial}}\big)$. Then the following hold:

(i) If $\varphi : X \to \mathbb{R}$ is Hölder continuous then φ_{req} is Hölder with singularities along Z_0, and φ_{eq} is Hölder with singularities along $\Sigma_1 \cup \ldots \cup \Sigma_\ell \cup Z_0$.

(ii) If $\varphi : X \to \mathbb{R}$ is continuous then φ_{req} is continuous on $X \setminus Z_0$, and φ_{eq} is continuous on $X \setminus (\Sigma_1 \cup \ldots \cup \Sigma_\ell \cup Z_0)$.

4. Convergence of Fubini-Study currents

Assume: X, L, Σ, τ verify (A)-(D), $\exists\,\omega$ Kähler form on X, h_0 is a fixed smooth Hermitian metric on L, h is a singular metric on L. Write

$$\alpha := c_1(L, h_0)\,,\ \ h = h_0 e^{-2\varphi}\,,\ \ \text{so } c_1(L,h) = \alpha + dd^c\varphi.$$

$\varphi \in L^1(X, \omega^n)$ is called the *(global) weight of h relative to h_0*.

h is called continuous, resp. Hölder continuous, if φ is as such on X.

$H^0_{(2)}(X, L^p) =$ *Bergman space of L^2-holomorphic sections of L^p* relative to the metric $h^p := h^{\otimes p}$ on L^p and volume ω^n on X

$$(S, S')_p := \int_X \langle S, S'\rangle_{h^p} \frac{\omega^n}{n!}\,,\ \ \|S\|_p^2 := (S,S)_p$$

We assume in the sequel that the metric h is **continuous** and consider

$$H^0_0(X, L^p) \subset H^0(X, L^p) = H^0_{(2)}(X, L^p)\,.$$

$\dim H^0_0(X, L^p) = d_p + 1, \quad S^p_0, \dots, S^p_{d_p}$ orthonormal basis of $H^0_0(X, L^p)$

$$P_p(x) = \sum_{j=0}^{d_p} |S^p_j(x)|^2_{h^p},\ x \in X \qquad \textit{(partial) Bergman kernel of } H^0_0(X, L^p)$$

Let $U \subset X$ open, such that L has a local holomorphic frame e_U on U:

$|e_U|_h = e^{-\varphi_U}$, $S^p_j = s^p_j e_U^{\otimes p}$, where $\varphi_U \in L^1_{loc}(U, \omega^n)$, $s_j \in \mathscr{O}_X(U)$.

$$\gamma_p\,|_U = \frac{1}{2}\, dd^c \log\Big(\sum_{j=0}^{d_p} |s^p_j|^2\Big) \qquad \textit{Fubini-Study current of } H^0_0(X, L^p)$$

Have: $\quad \log P_p\,|_U = \log\Big(\sum_{j=0}^{d_p} |s^p_j|^2\Big) - 2p\varphi_U, \quad$ so $\log P_p \in L^1(X, \omega^n)$

$$\tfrac{1}{p}\,\gamma_p = c_1(L, h) + \tfrac{1}{2p}\, dd^c \log P_p = \alpha + dd^c \varphi_p,$$

$$\varphi_p = \varphi + \tfrac{1}{2p} \log P_p = \textit{global Fubini-Study potential of } \gamma_p.$$

Note that φ_p is an α-psh function on X.

Theorem 8

Let X, L, Σ, τ verify assumptions (A)-(D), assume that (L, Σ, τ) is big and there exists a Kähler form ω on X. Let h be a continuous Hermitian metric on L. Then there exists an α-psh function φ_{eq} on X such that, as $p \to \infty$,

$$\int_X |\varphi_p - \varphi_{\mathrm{eq}}|\, \omega^n \to 0,\ \frac{1}{p}\gamma_p = \alpha + dd^c\varphi_p \to T_{\mathrm{eq}} := \alpha + dd^c\varphi_{\mathrm{eq}},$$

weakly on X. If h is Hölder continuous then $\exists\, C > 0$, $p_0 > 1$, such that

$$\int_X |\varphi_p - \varphi_{\mathrm{eq}}|\, \omega^n \leq C\,\frac{\log p}{p}\,,\ \textit{for all } p \geq p_0.$$

Definition 9

The current T_{eq} from Theorem 8 is called *the equilibrium current associated to* (L, h, Σ, τ).

Construction of $\varphi_{\rm eq}$: Let $(\widetilde{X},\pi,\widetilde{\Sigma})$ be a divisorization of (X,Σ) and set

$$\widetilde{L}:=\pi^\star L\,,\quad \widetilde{h}_0:=\pi^\star h_0\,,\quad \widetilde{\alpha}:=\pi^\star\alpha=c_1(\widetilde{L},\widetilde{h}_0),$$

$$\widetilde{\varphi}:=\varphi\circ\pi\,,\quad \widetilde{h}:=\pi^\star h=\widetilde{h}_0e^{-2\widetilde{\varphi}}\,.$$

Recall that $\quad H^0_0(X,L^p)=H^0_0(X,L^p,\Sigma,\tau,h^p,\omega^n)$.

The map

$$S\in H^0_0(X,L^p)\to\pi^\star S\in H^0_0(\widetilde{X},\widetilde{L}^p)=H^0_0(\widetilde{X},\widetilde{L}^p,\widetilde{\Sigma},\tau,\widetilde{h}^p,\pi^\star\omega^n)$$

is an isometry, so

$$\widetilde{P}_p=P_p\circ\pi\,,\quad \widetilde{\gamma}_p=\pi^\star\gamma_p\,,$$

are the Bergman kernel function, resp. Fubini-Study current, of $H^0_0(\widetilde{X},\widetilde{L}^p)$.

Have: $\quad \dfrac{1}{p}\,\widetilde{\gamma}_p=\widetilde{\alpha}+dd^c\widetilde{\varphi}_p, \quad$ where $\widetilde{\varphi}_p=\widetilde{\varphi}+\frac{1}{2p}\log\widetilde{P}_p=\varphi_p\circ\pi.$

Recall:

$$\mathcal{L}(\widetilde{X},\widetilde{\alpha},\widetilde{\Sigma},\tau)=\left\{\psi\in\mathrm{PSH}(\widetilde{X},\widetilde{\alpha}):\ \nu(\psi,x)\geq\tau_j,\ \forall\, x\in\widetilde{\Sigma}_j,\ 1\leq j\leq\ell\right\}$$

$$\widetilde{\varphi}_{\mathrm{eq}}=\widetilde{\varphi}_{\mathrm{eq},\widetilde{\Sigma},\tau}=\sup\left\{\psi:\ \psi\in\mathcal{L}(\widetilde{X},\widetilde{\alpha},\widetilde{\Sigma},\tau),\ \psi\leq\widetilde{\varphi}\ \text{on}\ \widetilde{X}\right\}$$

Have: $\widetilde{\varphi}_{\mathrm{eq}}\in\mathcal{L}(\widetilde{X},\widetilde{\alpha},\widetilde{\Sigma},\tau),\ \ \widetilde{\varphi}_{\mathrm{eq}}\leq\widetilde{\varphi}$ on X.

Fix a Kähler form $\widetilde{\omega}$ on $\widetilde{X}$ such that $\widetilde{\omega}\geq\pi^\star\omega$.

Theorem 10

In the setting of Theorem 8, we have $\widetilde{\varphi}_p\to\widetilde{\varphi}_{\mathrm{eq}}$ in $L^1(\widetilde{X},\widetilde{\omega}^n)$ as $p\to\infty$. If φ is Hölder continuous on X then there exist $C>0$, $p_0>1$, such that

$$\int_{\widetilde{X}}|\widetilde{\varphi}_p-\widetilde{\varphi}_{\mathrm{eq}}|\,\widetilde{\omega}^n\leq C\,\frac{\log p}{p}\,,\ \ \text{for all } p\geq p_0.$$

$(\widetilde{X}, \pi, \widetilde{\Sigma})$ divisorization of (X, Σ): $\exists\, Y \supset X_{\rm sing}$ an analytic subset of X, $\dim Y \leq n-2$, $E = \pi^{-1}(Y)$, $\pi : \widetilde{X} \setminus E \to X \setminus Y$ is a biholomorphism.

Define $\quad \varphi_{\rm eq} := \widetilde{\varphi}_{\rm eq} \circ \pi^{-1}$ on $X \setminus Y \subset X_{\rm reg}$. $\quad$ Then, as $p \to \infty$,

$$\int_{X\setminus Y} |\varphi_p - \varphi_{\rm eq}|\, \omega^n = \int_{\widetilde{X}\setminus E} |\widetilde{\varphi}_p - \widetilde{\varphi}_{\rm eq}|\, \pi^\star \omega^n \leq \int_{\widetilde{X}} |\widetilde{\varphi}_p - \widetilde{\varphi}_{\rm eq}|\, \widetilde{\omega}^n \to 0.$$

$\alpha + dd^c \varphi_{\rm eq} = \pi_\star(\widetilde{\alpha} + dd^c \widetilde{\varphi}_{\rm eq}) \geq 0$, so $\varphi_{\rm eq}$ is α-psh on $X \setminus Y$.

Since X is normal and $\dim Y \leq n-2$, we have that $\varphi_{\rm eq}$ extends to an α-psh function on X:

If $U \subset X$ is open and ρ is a smooth function on U with $dd^c\rho = \alpha$, then $v := \rho + \varphi_{\rm eq}$ is psh on $U \setminus Y$, hence it extends to a psh function on U.

5. Zeros of random sequences of holomorphic sections

Projectivization of spaces of holomorphic sections

$$\mathbb{X}_p := \mathbb{P}H^0_0(X, L^p)\,, \;\; d_p := \dim \mathbb{X}_p = \dim H^0_0(X, L^p) - 1\,, \;\; \sigma_p := \omega_{\mathrm{FS}}^{d_p}\,.$$

Product probability space $\quad (\mathbb{X}_\infty, \sigma_\infty) := \prod_{p=1}^{\infty} (\mathbb{X}_p, \sigma_p)$

Using the Dinh-Sibony equidistribution theorem for meromorphic transforms we obtain:

Theorem 11

Let X, L, Σ, τ verify (A)-(D) and h be a continuous Hermitian metric on L. Assume that (L, Σ, τ) is big and there exists a Kähler form ω on X. Then $\dfrac{1}{p}\,[s_p = 0] \to T_{\mathrm{eq}}$, as $p \to \infty$ weakly on X, for σ_∞-a.e. $\{s_p\}_{p\geq 1} \in \mathbb{X}_\infty$.

Theorem 12

Let X, L, Σ, τ verify (A)-(D), and h be a Hölder continuous Hermitian metric on L. Assume that (L, Σ, τ) is big and that there exists a Kähler form ω on X.

Then there exist constants $c > 0$, $p_0 > 1$, and subsets $E_p \subset \mathbb{X}_p$, such that for all $p \geq p_0$ we have:

(a) $\sigma_p(E_p) \leq cp^{-2}$,

(b) if $s_p \in \mathbb{X}_p \setminus E_p$ *we have*

$$\left|\left\langle \frac{1}{p}\,[s_p = 0] - T_{\rm eq}, \phi \right\rangle\right| \leq \frac{c \log p}{p} \, \|\phi\|_{\mathscr{C}^2} \,, \ \ \forall\, \phi \in \mathscr{C}^2_{n-1,n-1}(X).$$

In particular, the estimate from (b) *holds for* σ_∞*-a.e.* $\{s_p\}_{p\geq 1} \in \mathbb{X}_\infty$ *if* p *is large enough.*